

Hybrid meson masses and the correlated Gaussian basis

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We revisited a ar you model for charmonium hybrid meson with a magnetic gluon [Yu. S. Kalashnikova and A. V. Nefediev, *Phys. Rev. D* **77**, 054025 (2008).] and improved the numerical calculations. These improvements support the hybrid meson interpretation of $X(4260)$. Within the same model, we computed the hybrid meson mass with an electric gluon which is resolved to be lighter. Relativistic effects and coupling channels also decreased the mass.

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I. INTRODUCTION

Quantum Chromodynamics (QCD) is widely believed to be the theory of strong interactions. In this non-Abelian theory, the gauge bosons, the gluons, carry a color charge. We are therefore led to observable color singlet configurations made of quarks but also gluons. Glueballs, bound states of only gluons, and hybrid mesons are consequences and predictions of QCD. Besides glueballs, hybrid mesons deserve much interest. The gluonic excitation leads to low-lying states with quantum numbers not allowed for usual mesons (Note that the low-lying gluon states are nonexotic but glueballs with exotic quantum number also exists and are called oddballs; for a review see Ref. [1]). Their observation would be another great confirmation of QCD. However, the detection of exotic hadrons remains a challenging task (for a review see Ref. [2]).

The properties of hybrid mesons were investigated in various approaches (see Refs. [3,4] for examples), and in lattice QCD [5]. The former have two common interpretations. In the first one, the flux tube linking the quark to the antiquark is in an excited state, allowing quantum numbers that cannot be reached by the usual quark-antiquark picture. In the other scenario, the excitation is modeled by a constituent gluon, leading to a three-body system. We know that both pictures are closely linked: The constituent gluon creates an equivalent potential compatible with the energy of an excited string [6,7].

All these approaches rely on quasiparticle interpretation. For heavy particles, the spectrum and decay properties are extracted from a Hamiltonian formalism. One has then to resort to a numerical procedure to find eigenvalues and eigenfunctions. For two-body systems, like heavy charmonia, different methods exist. The most efficient one is the so-called Lagrange mesh method. This technique allows one to compute energies and wave functions straightforwardly since the method is not variational [8]. Matrix elements for semirelativistic kinetic energy are easily computed. However, it does not admit a three-body general-

ization in the semirelativistic case. For three-body systems, like hybrid mesons, one generally computes the mass within a finite dimensional basis. We expand the unknown wave function on a set of trial functions forming a basis of the Hilbert space in the limit where the dimension of the basis goes to infinity.

In practice, we only deal with a finite number of basis functions. Fortunately, we know the approximation of the mass to be always an upper of the true eigenvalue, allowing us to minimize the mass with respect to parameters [9]. The question is then: How many basis functions should we consider to have an acceptable accuracy? In this work, we answer to this question for the three-body description of hybrid mesons.

Recently, Kalashnikova and Nefediev investigated the spectrum of the charmonium hybrid $c\bar{c}g$ within the constituent gluon model [10]. They derived a Hamiltonian thanks to the field correlator method and introduced einbein (or auxiliary) fields to deal with relativistic kinematics. The resulting Hamiltonian was diagonalized with one trial function that was taken as a Gaussian depending of a hyperradius. Finally, the authors computed correction coming from the string, self-energy and spin-dependent operators. The authors studied the case of the lowest $c\bar{c}g$ hybrid meson in with the gluon quantum numbers are $\ell_g = 1$ and $j = 1$ (magnetic gluon). In this work, we investigate how accurate those approximations are for the wave function (truncation of the basis) and for the Hamiltonian (introduction of auxiliary fields) and greatly improve the accuracy on the mass thanks to correlated Gaussian functions.

In Sec. II, we present the model of Ref. [10] and recall its main properties. A particular ansatz was used for the wave function which is described in Sec. III. We discuss some improvements for the wave function in Sec. IV. In this section, which consists of the main part of this work, we perform numerical calculations with various approximations for the model and make some comparisons. Having identified a good approximation for the wave function, we test in Sec. V the stability of the error under parameter evolution. Finally, we draw our conclusion in Sec. VI

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II. HYBRID MESONS AS $q\bar{q}g$ SYSTEMS

The starting point for the hybrid model of [10] is a Hamiltonian derived from QCD thanks to the field correlator method. For a hybrid meson seen as a $q\bar{q}g$ system, this Hamiltonian reads

$$H = H_0 + V_C, \quad (1)$$

where

$$H_0 = \sqrt{\mathbf{p}_q^2 + m_q^2} + \sqrt{\mathbf{p}_{\bar{q}}^2 + m_{\bar{q}}^2} + \sqrt{\mathbf{p}_g^2} + \sigma|\mathbf{x}_g - \mathbf{x}_q| + \sigma|\mathbf{x}_g - \mathbf{x}_{\bar{q}}|, \quad (2)$$

$$V_C = -\frac{3\alpha_s}{2|\mathbf{x}_g - \mathbf{x}_q|} - \frac{3\alpha_s}{2|\mathbf{x}_g - \mathbf{x}_{\bar{q}}|} + \frac{\alpha_s}{6|\mathbf{x}_q - \mathbf{x}_{\bar{q}}|}. \quad (3)$$

The first terms of H_0 represent the kinetic energy of two quarks of the same mass and of the massless constituent gluon. The long-range confining force consists in two strings, two fundamental flux tubes of energy density σ , linking the gluon to the quarks. Such a confining potential is in agreement with other phenomenological approaches [11,12] as well as with recent lattice QCD computations [13]. For other three-body systems like baryons, the confinement is generally assumed to be a Y shape. In the case of the hybrid, however, under Casimir scaling hypothesis for the string tension, the V configuration is resolved to be energetically favorable [12]. The Torricelli point of the Y shape merges with the gluon position. Imposing a Y shape would obviously raise the whole spectrum. The Casimir scaling hypothesis is supported by recent lattice studies (see for instance Ref. [14]).

The confining potential is augmented by the short-range Coulomb potential V_C , where α_s is the strong coupling constant, arising from one gluon exchange effects between the constituent particles. In particular, the Coulomb potential between the quark and the antiquark is repulsive since the $q\bar{q}$ pair is in a color octet. We refer the reader to Ref. [15] for a detailed discussion about the short-range interaction potentials in QCD. Spin-dependent short-range interactions could be added in perturbation as it is done in Ref. [10]. However, it is enough for our purpose to consider only the dominant order Hamiltonian (1).

To be complete, we recall that the parity and charge conjugation of a $q\bar{q}g$ system are given, respectively, by

$$P = (-1)^{\ell_{q\bar{q}} + \ell_g}, \quad C = (-1)^{1 + \ell_{q\bar{q}} + S_{q\bar{q}}}, \quad (4)$$

where $\ell_{q\bar{q}}$ and ℓ_g are the orbital angular momenta of the $q\bar{q}$ pair and of the gluon, and where $S_{q\bar{q}}$ is the intrinsic spin of the $q\bar{q}$ pair. Compared to usual mesons, extra phases give rise to exotic quantum numbers such as 1^{-+} , for example. Two cases have to be distinguished following the total spin j of the gluon: The magnetic gluon, for which $\ell_g = j$, and the electric gluon for which $\ell_g = j \pm 1$ [10]. Although it may gain a dynamical mass induced by confining forces,

the gluon is a massless particle that remains transverse, with only two polarizations [1,16]. Therefore, the minimal value for j is 1. Exotic quantum numbers require at least one P wave. In the following, we shall refer to magnetic (electric), the $q\bar{q}g$ states with $\ell_g = 1$ ($\ell_g = 0$).

Solving the eigenequation associated to a three-body semirelativistic Hamiltonian such as (1) is a difficult numerical problem. The intrinsic complexity of three-body systems obviously comes into play, but another problem is the determination of the matrix elements for kinetic operators of the form $\sqrt{\mathbf{p}^2 + m^2}$, which are not commonly found in quantum mechanics. This last difficulty can be avoided by introducing auxiliary (or einbein) fields to get rid of the square roots appearing in Hamiltonian (1). One obtains

$$H(\mu_q, \mu_{\bar{q}}, \mu_g) = \frac{\mu_q + \mu_{\bar{q}} + \mu_g}{2} + \frac{\mathbf{p}_{\bar{q}}^2 + m_{\bar{q}}^2}{2\mu_q} + \frac{\mathbf{p}_q^2 + m_q^2}{2\mu_{\bar{q}}} + \frac{\mathbf{p}_g^2}{2\mu_g} + \sigma|\mathbf{x}_g - \mathbf{x}_q| + \sigma|\mathbf{x}_g - \mathbf{x}_{\bar{q}}| + V_C, \quad (5)$$

where μ_q , $\mu_{\bar{q}}$, and μ_g are the so-called auxiliary fields. Being formally defined as operators, they can be eliminated through the following equations:

$$\delta_{\mu_q} H(\mu_q, \mu_{\bar{q}}, \mu_g)|_{\mu_q = \hat{\mu}_q} = 0 \Rightarrow \hat{\mu}_q = \sqrt{\mathbf{p}_{\bar{q}}^2 + m_{\bar{q}}^2}, \quad (6)$$

$$\delta_{\mu_{\bar{q}}} H(\mu_q, \mu_{\bar{q}}, \mu_g)|_{\mu_{\bar{q}} = \hat{\mu}_{\bar{q}}} = 0 \Rightarrow \hat{\mu}_{\bar{q}} = \sqrt{\mathbf{p}_q^2 + m_q^2}, \quad (7)$$

$$\delta_{\mu_g} H(\mu_q, \mu_{\bar{q}}, \mu_g)|_{\mu_g = \hat{\mu}_g} = 0 \Rightarrow \hat{\mu}_g = \sqrt{\mathbf{p}_g^2}. \quad (8)$$

It is then readily checked that $H(\hat{\mu}_q, \hat{\mu}_{\bar{q}}, \hat{\mu}_g) = H_0$; both Hamiltonians are equivalent up to the elimination of the auxiliary fields as operators. However, the calculations are considerably simplified if one considers them as c-numbers variational parameters. The eigenvalues of the spinless Hamiltonian (5) are more easily found since only non-relativistic kinetic operators are present. These eigenvalues, denoted as $E_0(\mu_q, \mu_{\bar{q}}, \mu_g)$, are finally minimized with respect to the einbeins. The optimal values of the auxiliary fields seen as variational parameters are logically close to the average values of the corresponding operators [17,18]. For example, the optimal value of μ_g , denoted as μ_{g0} , is such that $\mu_{g0} \approx \langle \hat{\mu}_g \rangle$. It can be interpreted as a dynamical gluon mass. The same arguments hold for the other auxiliary fields. An important point, that has been shown in Ref. [19], is that the eigenvalues of Hamiltonian (5) are upper bounds of the eigenvalues of Hamiltonian (1). The more auxiliary fields are introduced, the less this bound is accurate. In particular, the accuracy of the auxiliary field method decreases when light or massless particles are present in the system under study.

In this work we focus on the case where the quark and the antiquark have the same mass, i.e. $m_q = m_{\bar{q}} = m$. Then, by symmetry, $\mu_q = \mu_{\bar{q}} = \mu$ and, using the Jacobi coordinates

$$\boldsymbol{\lambda} = \mathbf{x}_q - \mathbf{x}_{\bar{q}}, \quad \boldsymbol{\rho} = \mathbf{x}_g - \frac{\mathbf{x}_q + \mathbf{x}_{\bar{q}}}{2}, \quad (9)$$

Hamiltonian (5) becomes

$$H(\mu, \mu_g) = \mu + \frac{\mu_g}{2} + \frac{m^2}{\mu} + \frac{\mathbf{p}_\lambda^2}{\mu} + \frac{\mathbf{p}_\rho^2}{2\phi} + \sigma \left| \frac{\boldsymbol{\lambda}}{2} - \boldsymbol{\rho} \right| + \sigma \left| \frac{\boldsymbol{\lambda}}{2} + \boldsymbol{\rho} \right| + V_C, \quad (10)$$

where \mathbf{p}_λ and \mathbf{p}_ρ are the momenta associated to $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$, respectively, and where

$$\phi = \frac{2\mu\mu_g}{2\mu + \mu_g}. \quad (11)$$

The center of mass, defined as

$$\mathbf{R} = \frac{\mu\mathbf{x}_q + \mu\mathbf{x}_{\bar{q}} + \mu_g\mathbf{x}_g}{2\mu + \mu_g}, \quad (12)$$

is decoupled and its conjugate momentum is set equal to zero since we work in the rest frame of the system.

III. HYPERSPHERICAL FORMALISM

Eigenvalues of Hamiltonian $H(\mu, \mu_g)$ with auxiliary fields (10) in the case of $c\bar{c}g$ systems have been numerically computed in Ref. [10] by using the hyperspherical formalism. We recall in this section the procedure that has been used in this last reference. The authors of [10] state that since the $q\bar{q}$ pair is heavy, the assumption $\ell_{q\bar{q}} = 0$ can be made. Moreover, the particular case of a magnetic gluon with $\ell_g = 1$ is considered in this reference. For this particular choice, the spin of the quark pair $S_{q\bar{q}}$ is a good quantum number and leads to the following states: 1^{--} for $S_{q\bar{q}} = 0$ and $(0, 1, 2)^{-+}$ for $S_{q\bar{q}} = 1$. All these states are degenerate since $H(\mu, \mu_g)$ does not have any spin-dependent term. This leads to wave functions of the form

$$|\chi(\mathbf{X})\rangle \otimes [|\ell_{q\bar{q}} = 0, \ell_g = 1\rangle^1 \otimes |S_{q\bar{q}}\rangle]^J, \quad (13)$$

with the hyperradius

$$X^2 = \frac{\mu}{2} \boldsymbol{\lambda}^2 + \frac{2\mu\mu_g}{2\mu + \mu_g} \boldsymbol{\rho}^2. \quad (14)$$

$|\ell_{q\bar{q}}, \ell_g\rangle^\ell$ is a shorthand notation for the coupling $[\mathcal{Y}_{\ell_{q\bar{q}}}(\boldsymbol{\lambda}) \otimes \mathcal{Y}_{\ell_g}(\boldsymbol{\rho})]^\ell$, with the solid spherical harmonics $\mathcal{Y}_{\ell_g m}(\boldsymbol{\rho}) = \rho^{\ell_g} Y_{\ell_g m}(\hat{\boldsymbol{\rho}})$. The following states can consequently be described within this approach

$$|1^{--}\rangle = |\chi(\mathbf{X})\rangle \otimes [|[0, 1]^1 \otimes |0\rangle]^1 \quad (15)$$

$$|J^{-+}\rangle = |\chi(\mathbf{X})\rangle \otimes [|[0, 1]^1 \otimes |1\rangle]^J, \quad (16)$$

with $J = \{0, 1, 2\}$. The trial wave function is chosen to be a Gaussian depending on one variational parameter β , i.e.

$$\langle \boldsymbol{\rho}, \boldsymbol{\lambda} | [\chi(\mathbf{X})\rangle \otimes |0, 1\rangle^1] = \exp\left(-\frac{1}{2}\beta^2 X^2\right) \mathcal{Y}_{1m}(\boldsymbol{\rho}). \quad (17)$$

This solid harmonic, $\mathcal{Y}_{1m}(\boldsymbol{\rho}) = \rho Y_{1m}(\hat{\boldsymbol{\rho}})$, determines the angular momentum in order to treat exotic mesons with a P wave gluon and an S wave $q\bar{q}$ pair. The spin function for the quark pair is irrelevant since our Hamiltonian is spin-independent. We will therefore get the same mass for the four states 0^{-+} , 1^{--} , 1^{-+} , 2^{-+} under consideration.

A convenient method to fit the parameters is to reproduce the charmonium spectrum within the same flux tube model, that is with the Hamiltonian $H_{c\bar{c}} = 2\sqrt{\mathbf{p}^2 + m_c^2} + \sigma r - 4\alpha_s/3r$. The parameters obtained in this way are compatible with typical values used in potential models [10]:

$$\sigma = 0.16 \text{ GeV}^2, \quad \alpha_s = 0.55, \quad m_c = 1.48 \text{ GeV}. \quad (18)$$

We, as did the authors of Ref. [10], then implicitly assumed the same value for the parameters for both systems (charmonium and hybrid charmonium). It is also natural to keep the same parameters as in Ref. [10] since the present paper focuses on the improvement of the numerical resolution. In Sec. V we vary the parameters and investigate this influence of the one Gaussian approximation.

A numerical resolution of $H(\mu, \mu_g)$ with the ansatz (17) leads to [10]

$$M_0 = 4.573 \text{ GeV}, \quad \mu_0 = 1.598 \text{ GeV}, \quad \mu_{g0} = 1.085 \text{ GeV}, \quad (19)$$

for the ground state. This mass corresponds in a first approximation to $c\bar{c}$ hybrid mesons with quantum numbers 1^{--} and $(0, 1, 2)^{-+}$.

Spin-dependent corrections have also been computed to be equal to [10]

$$\Delta M(0^{-+}) = -321 \text{ MeV}, \quad (20a)$$

$$\Delta M(1^{-+}) = -253 \text{ MeV}, \quad (20b)$$

$$\Delta M(1^{--}) = -176 \text{ MeV}, \quad (20c)$$

$$\Delta M(2^{-+}) = -116 \text{ MeV}. \quad (20d)$$

The splittings induced by the spin-dependent operators in perturbation theory are clearly not negligible. However, we know from the Rayleigh-Ritz method [9] that the mass $M_0 = 4.573 \text{ GeV}$ can only be an upper bound of the true eigenvalue of $H(\mu, \mu_g)$. Indeed, when truncating the basis with a finite number of basis function, one restrict the operator to the subspace spanned by the trial functions. It is then natural to investigate how far is M_0 from the true eigenvalue and compare the difference with respect to the additional corrections (20).

IV. CORRELATED GAUSSIAN BASIS

The real eigenvalue M of a Schrödinger equation $H|\Phi\rangle = M|\Phi\rangle$ is generally found by expanding the wave function $\Phi(\boldsymbol{\lambda}, \boldsymbol{\rho}) = \langle \boldsymbol{\rho}, \boldsymbol{\lambda} | \Phi \rangle$ in a basis

$$\Phi(\boldsymbol{\lambda}, \boldsymbol{\rho}) = \sum_i^N \alpha_i \varphi^i(\boldsymbol{\lambda}, \boldsymbol{\rho}). \quad (21)$$

The real eigenvalue is reached in the limit $N \rightarrow \infty$. Since for fixed N , $M^{(N)}$ is an upper bound, we can then minimize the mass with respect to the parameters on which depend φ^i . An appropriate ansatz for φ^i will lead to an accurate mass for low values of N .

For the trial functions, we use a generalization of the hyperspherical function (17)

$$\begin{aligned} \varphi^i(\boldsymbol{\lambda}, \boldsymbol{\rho}) &= \exp(-a_i \boldsymbol{\lambda}^2 - b_i \boldsymbol{\rho}^2 - 2c_i \boldsymbol{\lambda} \cdot \boldsymbol{\rho}) \\ &\times \mathcal{Y}_{LM}(d_i \boldsymbol{\lambda} + e_i \boldsymbol{\rho}). \end{aligned} \quad (22)$$

The wave function used in [10] is recovered for the particular values $N = 1$, $c_1 = d_1 = 0$ and the condition (14) on the variational parameters a_1 and b_1 . The correlated Gaussian (22) shared many advantages [9]. They admit a generalization for an arbitrary number of interacting particles thanks to matrix notation; matrix elements for usual power laws potential are expressed in close forms; the Fourier transform of a Gaussian is also a Gaussian. In particular, we have shown that matrix elements for semi-relativistic kinematics $\sqrt{\boldsymbol{p}^2 + m^2}$ can be easily derived [20,21].

This form is very convenient for hybrid mesons since we deal easily with angular excitations for the gluon but also for the quarks. Imposing some restriction on the parameters we get two ansatz for the lowest P wave hybrid mesons

$$\varphi_A(\boldsymbol{\lambda}, \boldsymbol{\rho}) = \exp(-a \boldsymbol{\lambda}^2 - b \boldsymbol{\rho}^2) \mathcal{Y}_{1m}(\boldsymbol{\rho}), \quad (23)$$

$$\varphi_B(\boldsymbol{\lambda}, \boldsymbol{\rho}) = \exp(-a \boldsymbol{\lambda}^2 - b \boldsymbol{\rho}^2) \mathcal{Y}_{1m}(\boldsymbol{\lambda}). \quad (24)$$

The function $\varphi_A(\boldsymbol{\lambda}, \boldsymbol{\rho})$ has the angular part $|\ell_{q\bar{q}} = 0, \ell_g = 1\rangle^1$ and corresponds to a hybrid meson with a magnetic gluon, and the function $\varphi_B(\boldsymbol{\lambda}, \boldsymbol{\rho})$ has the angular behavior $|\ell_{q\bar{q}} = 1, \ell_g = 0\rangle^1$ and corresponds to a hybrid meson with an electric gluon.

A. P wave hybrids with magnetic gluon

Let us now see how the mass changes when relaxing the condition (14) on a_1 and b_1 . In order to have a relevant comparison point, we consider $H(\mu, \mu_g)$ with the parameters given in (18) and (19). If we use the hyperspherical formalism with only one trial function (and hence only one variational parameter) in our basis, we find a mass of $M_0 = 4.573$ GeV (obviously the same value as in Ref. [10]). The optimization procedure leads to $\beta = 0.62$ GeV^{1/2} which corresponds to $a^H = 0.154$ GeV and $b^H = 0.156$ GeV. The relation (14) with the parameters (18) implies a nearly

symmetric Gaussian shape, i.e

$$\frac{b^H}{a^H} = \frac{2\mu + \mu_g}{4\mu_g} = 1.0138. \quad (25)$$

Here we perform a numerical optimization on the two parameters of our single wave function $\varphi_A(\boldsymbol{\lambda}, \boldsymbol{\rho})$. We expect to find a mass slightly lower than in [10]. Indeed, the resulting mass for the four states 0^{-+} , 1^{-+} , 1^{-+} , 2^{-+} in this approximation reads

$$M_A^{(1)} = 4.462 \text{ GeV}. \quad (26)$$

We gain 109 MeV by simply choosing a more general ansatz (with two variational parameters instead of one parameter) for the trial wave function. The shift is of order of the spin splittings (20). Let us look at the wave in order to understand this difference. The optimization procedure leads to $a_1 = 0.083$ GeV and $b_1 = 0.204$ GeV. Their ratio strongly differs from (25). As an illustration, we plot the spatial part of the wave functions in Figs. 1 and 2 (recall the factor ρ in the solid spherical harmonic).

We can go even further by adding more functions in the basis. With a sufficient number of functions we should converge to the real eigenvalue of our operator $H(\mu, \mu_g)$. Convergence is obtained for a small number of Gaussians. With $N = 10$ functions, the numerical procedure is accurate up to 1 MeV which is clearly enough. The wave functions are then a sum of 10 Gaussian trial functions. The coefficients and parameters of each of the functions can be read in Table I. The resulting mass,

$$M_A^{(10)} = 4.445 \text{ GeV}, \quad (27)$$

is somewhat lower. The gain when increasing the basis is not as strong as the first gain obtained with one trial Gaussian (23) with two variational parameters. We can

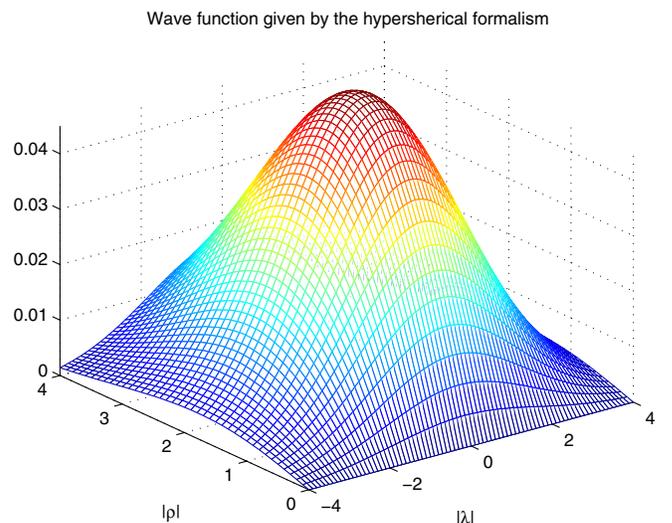


FIG. 1 (color online). Wave function (17) for a magnetic gluon hybrid meson.

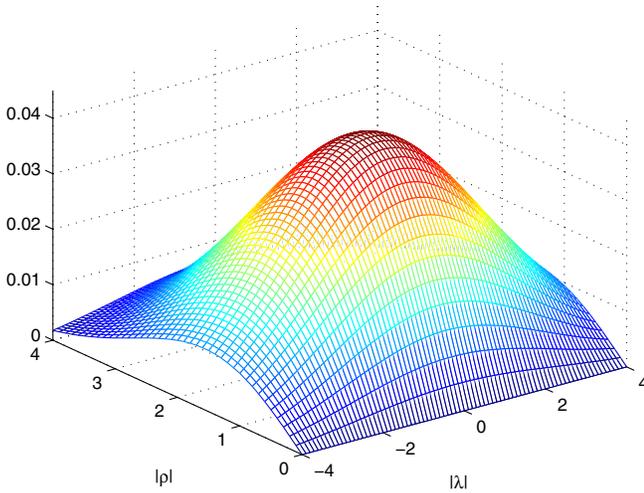
Wave function with $N = 1$ function in the basis

FIG. 2 (color online). Wave function (23) for a magnetic gluon hybrid meson.

understand this effect as follows: The Gaussian wave function (23) has a good overlap with the exact wave function of the three-body system. However, when we restrict the trial function to be a Gaussian of the only hyperradius (17), the overlap with the wave function is worst since the hyperradius imposes a relation between the two coefficients of Jacobi variables which is not realized in the optimal case. The wave functions with $N = 10$ functions are shown in Fig. 3. The relative error between the first trial function (17) and the eigenfunction is 33% and the relative error (i.e. the ratio between the integral of the difference squared and the integral of the function with $N = 10$ squared) with the single Gaussian approximation is 18%. Nevertheless, the mass found with one hyperspherical Gaussian is less than 3% above the real value.

In their paper, the authors of [10] concluded favorably in a hybrid meson interpretation of the vector meson candidate $X(4260)$. This new state was observed by the *BABAR* collaboration [22]. But they mentioned also that their mass $M_0 + \Delta M(1^{--}) = 4.397$ GeV was higher than the candidate's mass. It is worth mentioning that the exact value of

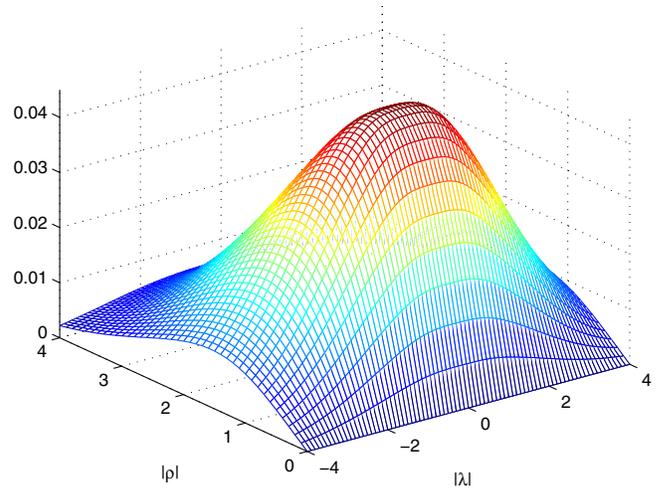
Wave function with $N = 10$ functions in the basis

FIG. 3 (color online). Wave function (22) with 10 Gaussian functions (23) for a magnetic gluon hybrid meson.

the Hamiltonian $H(\mu, \mu_g)$ used in [10] is 130 MeV below the approximative value computed with the trial function (20). Assuming that spin-dependent corrections do not change much with the change of the wave function, we get a mass compatible with the $X(4260)$.

B. P wave hybrids with electric gluon

In the previous section, we focused on the magnetic gluon hybrids. We followed the approach of [10] where the authors assumed an S wave quark-antiquark pair. Since, the quark-antiquark pair is a repulsive octet state, one can then wonder if it would not be energetically favorable to consider a P wave for the $q\bar{q}$ instead of exciting the gluon. The trial function respecting this assumption is $\varphi_B(\boldsymbol{\rho}, \boldsymbol{\lambda})$ in (24). This ansatz ($\ell_{q\bar{q}} = 1$), not investigated in Ref. [10], corresponds to an electric gluon hybrid with quantum numbers J^{--} , $J \in [0, 3]$ or J^{+-} , $J \in [0, 2]$.

First, we used only one single trial function depending on the hyperradius. We considered the same Hamiltonian H with the same value for σ , α_s , m_c , μ , and μ_g . We found a mass $M_0' = 4.225$ GeV which is already lower than the mass of a magnetic gluon hybrid. This time, the variational parameter has the value $\beta = 0.633$ GeV^{1/2}.

As a next step, we diagonalized $H(\mu, \mu_g)$ with one Gaussian trial function with two parameters. We found a mass $M_B^{(1)} = 4.137$ GeV with only one Gaussian function $\varphi_B(\boldsymbol{\rho}, \boldsymbol{\lambda})$. The optimal values for the parameters read: $a_1 = 0.119$ GeV and $b_1 = 0.232$ GeV. As in the case of a magnetic gluon, we found a mass roughly 100 MeV lower than with one function with the hyperspherical formalism. We plot the spatial part of the wave functions in Figs. 4 and 5 (recall the factor λ in the solid spherical harmonic).

The next step is obviously to check how accurate is the one Gaussian approximation by looking for the exact

TABLE I. Wave function coefficients for magnetic gluon (23).

i	$\alpha_i 10^{-3}$	a_i	b_i
1	26.183	0.082 811	0.203 578
2	116.410	0.207 870	0.697 459
3	-6.902	0.435 376	0.232 531
4	-3.242	2.429 518	0.720 520
5	-4.284	0.050 134	0.088 911
6	-11.033	0.079 373	1.083 885
7	46.785	0.153 637	0.383 596
8	8.5083	0.053 970	0.097 767
9	-0.505	0.165 116	0.078 640
10	-98.834	0.208 914	0.571 826

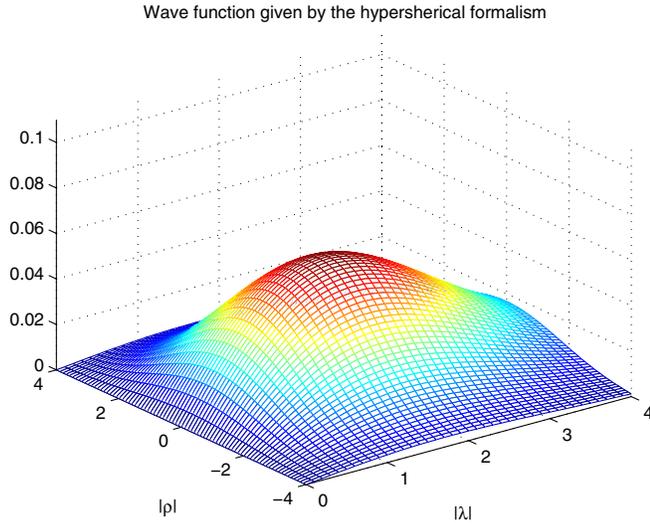


FIG. 4 (color online). Wave function (17) for an electric gluon hybrid meson.

eigenvalue of $H(\mu, \mu_g)$ (by exact, we mean a value stable at 1 MeV). When increasing N , the dimension of our basis, we reach a stable mass from $N = 10$. We check up to $N = 30$ the stability of our eigenvalue. The mass of electric gluon hybrid reads

$$M_B^{(10)} = 4.115 \text{ GeV}. \quad (28)$$

The coefficients and parameters of each of the functions can be read in Table II. Once again, the difference between the simple Gaussian and the true is five times lower than the difference between the two first approximations. The wave functions for $N = 10$ is shown in Fig. 6. The relative error between the first trial function (17) and the eigenfunction is 29% and the relative error with the single Gaussian approximation is 26%.

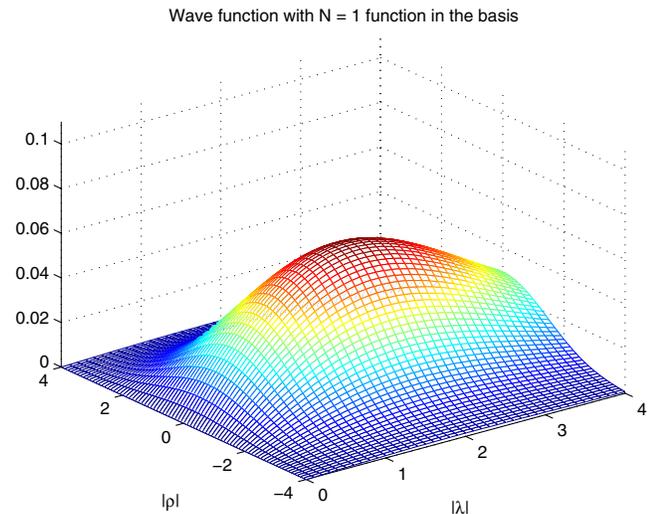


FIG. 5 (color online). Wave function (23) for an electric gluon hybrid meson.

TABLE II. Wave function coefficients for electric gluon

i	$\alpha_i 10^{-3}$	a_i	b_i
1	38.013	0.119 838	0.232 252
2	0.948	0.051 007	0.087 008
3	-0.977	0.039 315	0.393 460
4	5.333	0.319 606	0.714 934
5	-18.77	0.091 608	1.320 736
6	-5.593	0.398 742	0.209 574
7	9.742	0.070 955	1.500 372
8	55.451	0.257 114	0.710 774
9	2.330	0.066 225	0.219 531
10	34.224	1.050 299	2.419 412

We then conclude that the lowest exotic hybrid mesons 0^{--} , 1^{-+} , 3^{-+} are dominated by the electric gluon component.

C. P wave hybrid meson

We have identified two main components in the lowest P wave hybrid mesons. The electric gluon component was resolved to be lighter than the magnetic gluon component since the quarks are in an octet state. The physical wave function should be a mixing of those two situations since our Hamiltonian couples the two configurations. We expect then a lower mass for the ground state since coupling two channels repels each other. The more general ansatz for a P wave hybrid meson is

$$\Phi(\boldsymbol{\lambda}, \boldsymbol{\rho}) = \sum_i^N \alpha_i \exp(-a_i \boldsymbol{\lambda}^2 - b_i \boldsymbol{\rho}^2 - 2c_i \boldsymbol{\lambda} \cdot \boldsymbol{\rho}) \times \mathcal{Y}_{1M}(d_i \boldsymbol{\lambda} + e_i \boldsymbol{\rho}), \quad (29)$$

and corresponds to 0^{-+} , 1^{--} , 1^{-+} , 2^{-+} . Using this ansatz, we indeed find a lowest value for the ground state of $H(\mu, \mu_g)$:

$$M = 4.068 \text{ GeV}. \quad (30)$$

The resulting mass is indeed lower than in the previous cases.

D. Semirelativistic Hamiltonian

We showed in the previous sections that the ansatz for the wave function is essential to find the correct eigenvalues of an operator. One simple exponential of the hyper-radius may lead to a mass around 100 MeV above the true value. But, in this work, we assume that the correct Hamiltonian for the $q\bar{q}g$ system is H since the einbein are operators and not numbers. As a consequence, the eigenvalue of $H(\mu, \mu_g)$ are upper bounds of eigenvalues of H . The auxiliary fields were introduced to get rid of the square roots in H_0 . Treating μ and μ_g as ordinary numbers instead of operators causes an error on the energy around 5% [23]. Let us also note that, the more einbein there are,

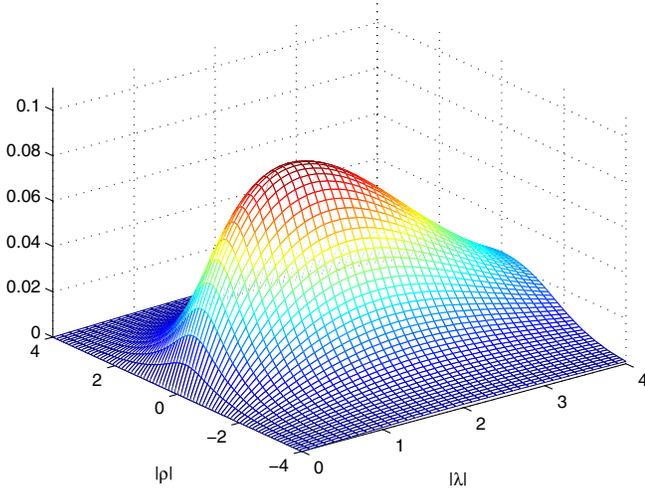
Wave function with $N = 10$ functions in the basis

FIG. 6 (color online). Wave function (22) with 10 Gaussian functions (23) for an electric gluon hybrid meson.

the less accurate the approximation is. We can avoid this approximation by diagonalizing H given by (1).

The Fourier transform of the general trial function (22) has also a Gaussian shape. We can then easily compute matrix elements for an operator in momentum space such as the semirelativistic kinetic operator [20]. With a basis of Gaussian functions, we do not need to introduce auxiliary fields. This also avoids determining the optimal values of μ and μ_g .

We are in the position to find the accurate eigenvalues of the semirelativistic Hamiltonian H (1). We computed the masses of the lightest hybrid with magnetic (φ_A) and electric (φ_B) gluons with $N = 1$ and $N = 10$ functions in the basis. The masses are below the ones obtained with the auxiliary field approximation and read

$$M_{0A}^{(1)} = 4.402 \text{ GeV}, \quad M_{0B}^{(1)} = 4.017 \text{ GeV}, \quad (31)$$

$$M_{0A}^{(10)} = 4.361 \text{ GeV}, \quad M_{0B}^{(10)} = 3.970 \text{ GeV}. \quad (32)$$

The coefficients and parameters of each function can be read in Table III for a magnetic gluon and in Table IV for an electric gluon. The wave functions are also displayed in Figs. 7 and 8 for a magnetic and electric gluon, respectively. The relative difference between the real values of H and $H(\mu, \mu_g)$ is less than 4%. With the wave functions, we computed the expectation values (for $N = 20$)

$$\mu_0 = \langle \Phi | \sqrt{p_q^2 + m_c^2} | \Phi \rangle, \quad \mu_{g0} = \langle \Phi | \sqrt{p_g^2} | \Phi \rangle. \quad (33)$$

In both cases, magnetic (A) and electric (B) gluons, the expectation values of these operators are compatible with the parameters (19) of [10]:

$$\mu_0^A = 1.652 \text{ GeV}, \quad \mu_{g0}^A = 0.976 \text{ GeV}, \quad (34)$$

TABLE III. Wave function coefficients for magnetic gluon with the semirelativistic Hamiltonian.

i	$\alpha_i 10^{-2}$	a_i	b_i
1	3.138	0.086 588	0.205 722
2	6.262	0.216 618	0.781 620
3	0.179	0.055 116	0.070 758
4	-0.287	0.608 667	0.201 197
5	108.510	0.875 692	0.739 526
6	3.673	0.161 373	0.426 376
7	-0.742	0.059 422	1.098 722
8	-2.084	0.339 798	0.350 189
9	39.117	1.054 584	0.569 019
10	-145.020	0.939 776	0.679 936

TABLE IV. Wave function coefficients for electric gluon with the semirelativistic Hamiltonian.

i	$\alpha_i 10^{-2}$	a_i	b_i
1	3.672	0.125 160	0.214 905
2	0.130	0.059 228	0.058 032
3	89.281	0.342 251	0.796 071
4	3.431	0.580 957	0.262 103
5	-0.666	0.150 242	1.634 497
6	-8.155	0.491 529	0.358 052
7	21.727	1.401 183	3.854 743
8	-75.356	0.341 822	0.825 713
9	-0.076	0.085 946	0.747 611
10	-3.160	2.586 533	0.989 884

$$\mu_0^B = 1.736 \text{ GeV}, \quad \mu_{g0}^B = 0.750 \text{ GeV}. \quad (35)$$

As expected, $\mu_0^A < \mu_0^B$ and $\mu_{g0}^A > \mu_{g0}^B$ since in the magnetic case, the P wave is reported on the gluon.

Wave function for the semi-relativistic Hamiltonian

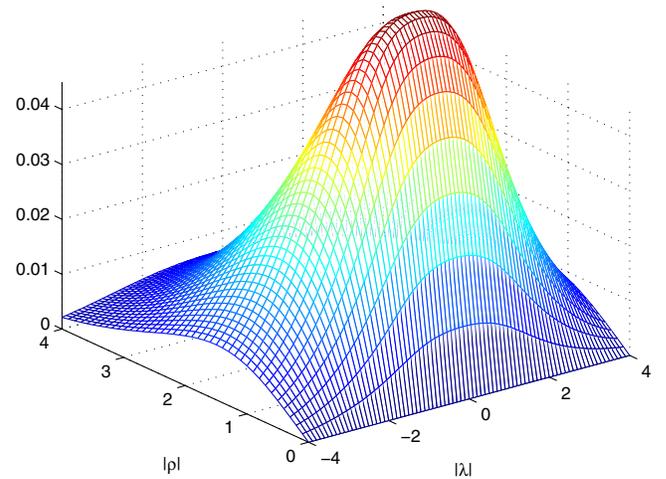


FIG. 7 (color online). Wave function for the semirelativistic Hamiltonian (22) with 10 Gaussian functions (23) for a magnetic gluon hybrid meson.

Wave function for the semi-relativistic Hamiltonian

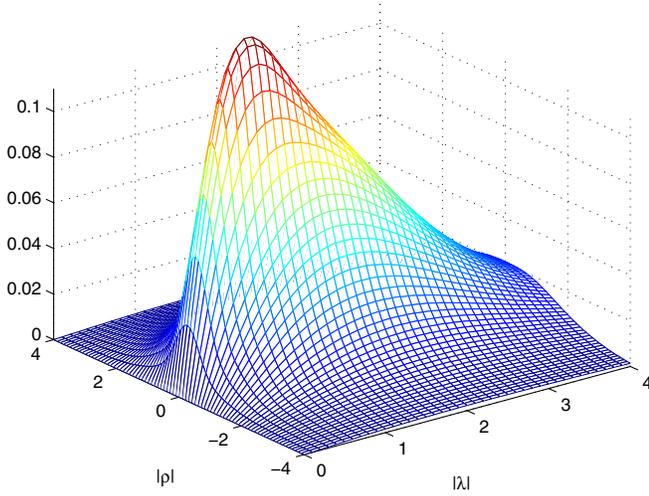
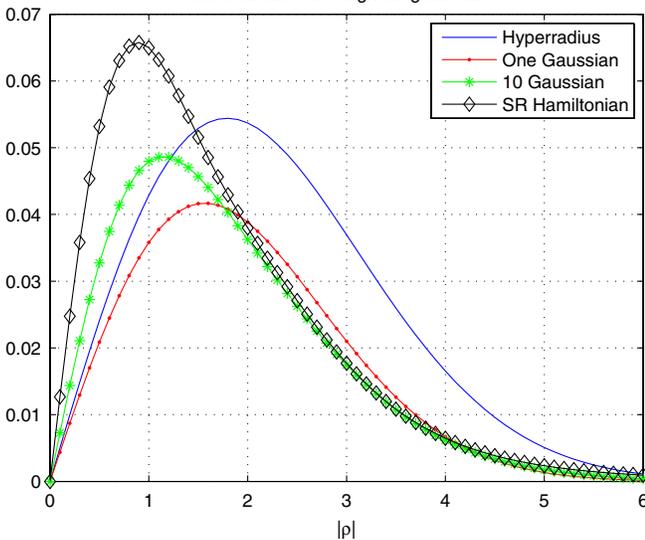
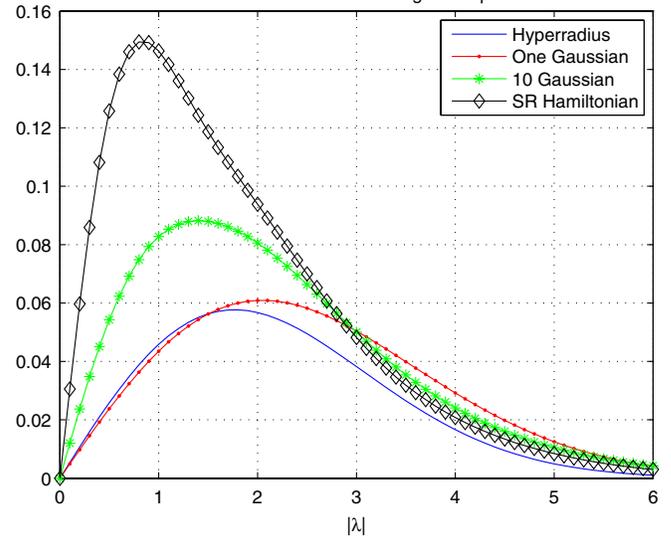


FIG. 8 (color online). Wave function for the semirelativistic Hamiltonian (22) with 10 Gaussian functions (23) for an electric gluon hybrid meson.

TABLE V. Spectra of (10) and (1) with different wave functions. Parameters (18) are used. All masses are in GeV.

	Magn. (A)	Elec. (B)	Full
$H(\mu, \mu_g), \chi(\mathbf{X})$ (17)	4.573	4.225	
$H(\mu, \mu_g), N = 1$	4.462	4.137	4.137
$H(\mu, \mu_g), N \rightarrow \infty$	4.445	4.115	4.068
$H, N = 1$	4.402	4.017	4.017
$H, N \rightarrow \infty$	4.361	3.970	3.895

Wave function for magnetic gluon at $\lambda=0$ FIG. 9 (color online). Wave function for a magnetic gluon at $\lambda = 0$.Wave function for electric gluon at $\rho=0$ FIG. 10 (color online). Wave function for an electric gluon at $\rho = 0$.

The final step is, of course, to diagonalize H without any assumption on the wave function, i.e. with (29). The ground state mass decreases once again and we obtain

$$M_0 = 3.895 \text{ GeV.} \quad (36)$$

We summarize all results in Table V, namely, the spectra of $H(\mu, \mu_g)$ and H for magnetic (A) and electric (B) gluons, but also for a more general P wave $c\bar{c}g$ hybrid mesons (Full). In each case, we displayed the results for $N = 1$ trial Gaussian functions and the true values obtained typically with $N = 10$ trial functions. In the general P wave cases, the $N = 1$ give always the electric gluon mass since the latter is the main component of the wave function. In view of this Table, we notice a gain of 600 MeV when relaxing all hypotheses.

Wave function given by the hyperspherical formalism

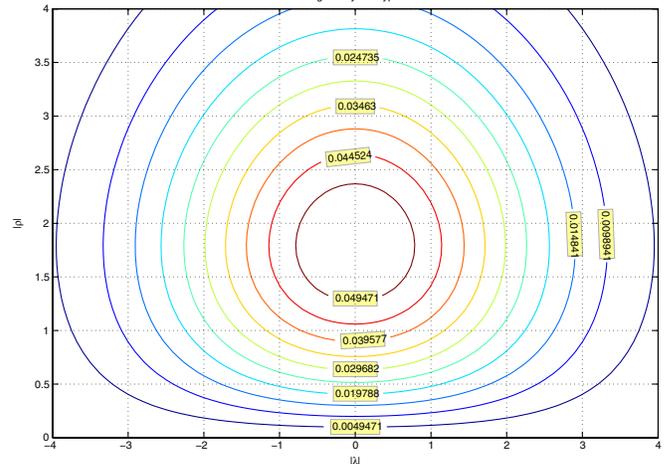


FIG. 11 (color online). Wave function (17) for a magnetic gluon hybrid meson.

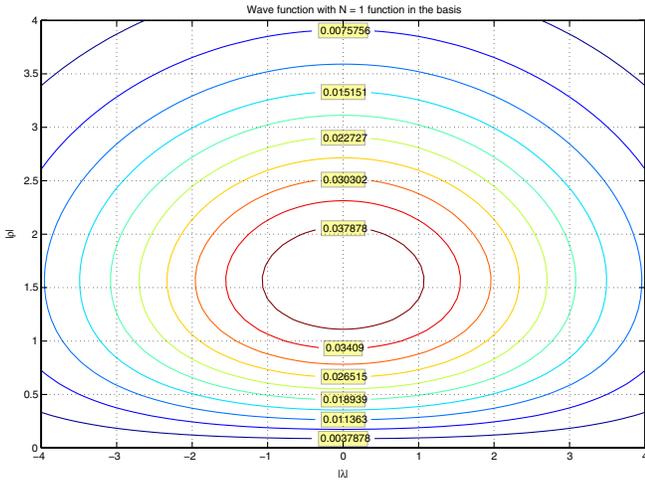


FIG. 12 (color online). Wave function (23) for a magnetic gluon hybrid meson.

In order to compare qualitatively the different wave functions, we display in Fig. 9 the wave functions for a magnetic gluon at $\lambda = 0$ and in Fig. 10 the wave functions for an electric gluon at $\rho = 0$.

For the sake of completeness and in order to compare all the wave functions quantitatively, we add the contour lines of all the eight wave functions described in the text. Those plots are displayed in Figs. 11–18.

The mass of the hybrid with an electric gluon is resolved to be lighter than the magnetic one. Our simple Hamiltonian mixes the two states and the mass of the lightest states has clearly a bigger electric component. However, the decay modes of these two states are different. An electric gluon allows the hybrid to decay into an S wave $D^* \bar{D}^*$. This may induce difficulty to single out a hybrid interpretation with respect to conventional charmonium. It would be interesting to investigate more deeply the decay properties of those states which could guide the experimentalist in the search for hybrid charmonia.

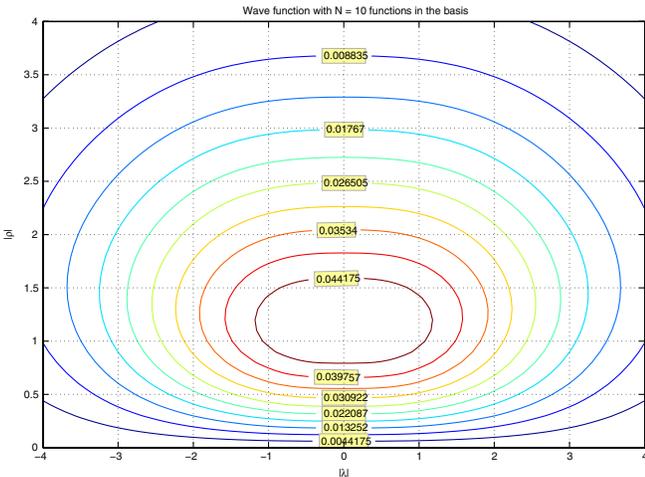


FIG. 13 (color online). Wave function (22) with 10 Gaussian functions (23) for a magnetic gluon hybrid meson.

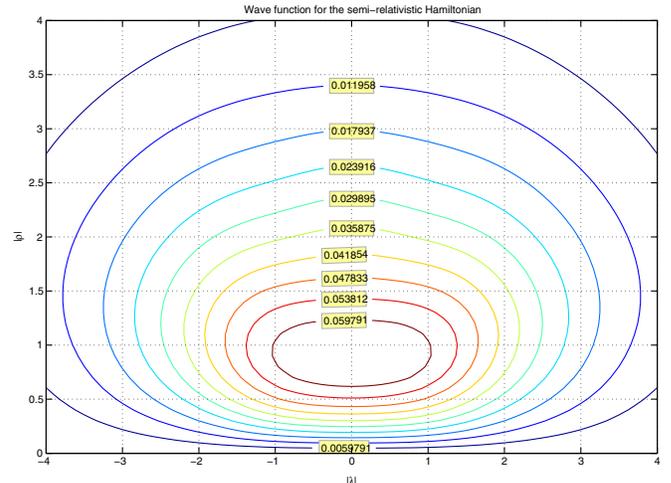


FIG. 14 (color online). Wave function for the semirelativistic Hamiltonian (22) with 10 Gaussian functions (23) for a magnetic gluon hybrid meson.

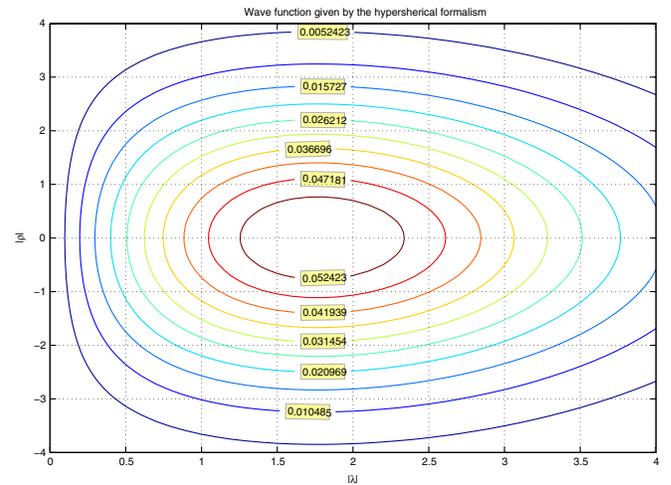


FIG. 15 (color online). Wave function (17) for an electric gluon hybrid meson.

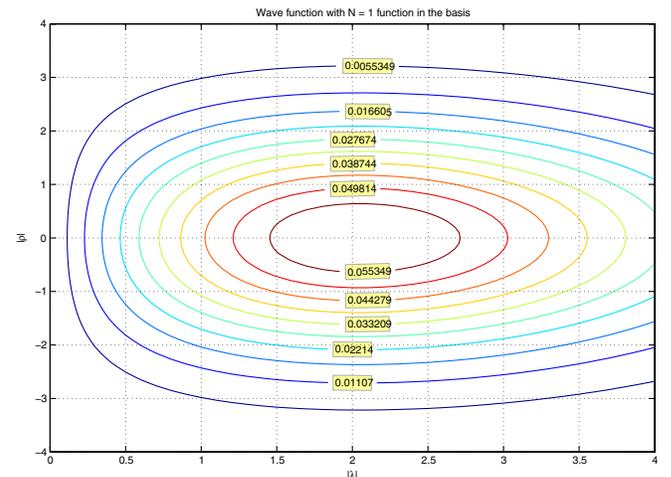


FIG. 16 (color online). Wave function (23) for an electric gluon hybrid meson.

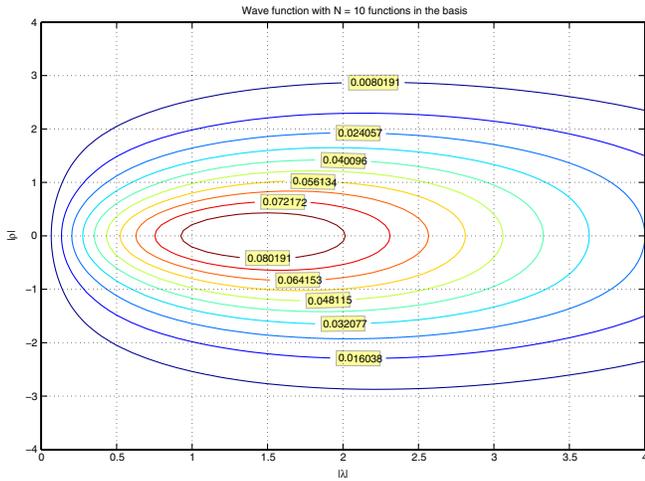


FIG. 17 (color online). Wave function (22) with 10 Gaussian functions (23) for an electric gluon hybrid meson.

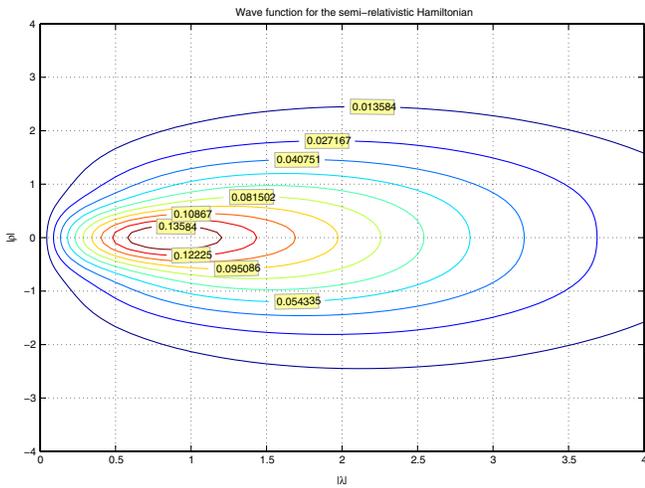


FIG. 18 (color online). Wave function for the semirelativistic Hamiltonian (22) with 10 Gaussian functions (23) for an electric gluon hybrid meson.

V. PARAMETER INFLUENCE

The previous sections emphasized the fact that various approximations may lead to an overestimation of hybrid masses. However, the single Gaussian approximation was resolved to be an acceptable approximation. We now turn our attention to this specific approximation and test its evolution for different values of the parameters for the semirelativistic Hamiltonian (1). We then change one parameter and keep the others to their “optimal” values (18). The results are displayed in Figs. 19 and 20 for, respectively, the evolution of the gluon mass and the string tension. Varying the gluon mass is interesting since effective approaches used a nonvanishing value for the gluon mass in the kinetic energy (see for instance Ref. [7]). For wide ranges of the parameters, the single Gaussian approximation is resolved always to give an overestimation of ~ 40 MeV constant in the intervals. For a wide range of

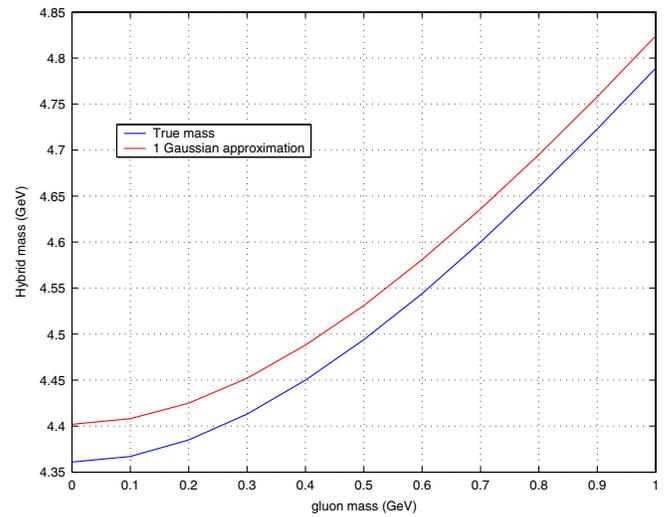


FIG. 19 (color online). Hybrid mass evolution according to gluon mass.

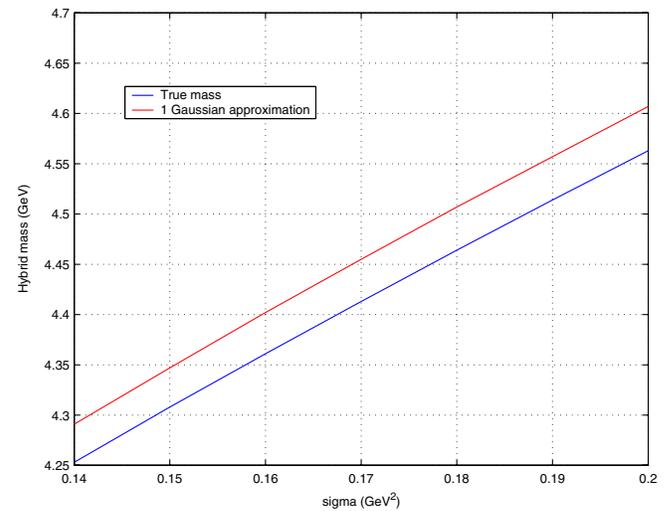


FIG. 20 (color online). Hybrid mass evolution according to string tension.

variation of α , the single Gaussian approximation lie around 20–50 MeV above the true mass. For hybrid meson systems, the approximative wave function (26) is then robust under parameter evolution. We checked that we obtain similar results for electric gluon wave functions.

VI. CONCLUSION

During the main part of this paper, we were concerned by Kalashnikova and Nefediev’s model [10] in which the starting point was Hamiltonian (1). This model shares the constituent model’s typical features. The kinetic energy is the semirelativistic expression valid for massless particles (the constituent gluon). A linear (in fact, a V junction) plus Coulomb term is used for the potential. The parameters (charm quark mass, string tension, and strong coupling) were determined on a similar model for charmonium. The

parameters (18) reproduce a quarkonium spectrum in agreement with experimental data and are expected to predict an acceptable value for hybrid masses.

However, solving three-body systems may require hypotheses: On one side in the Hamiltonian by introducing auxiliary fields to get rid of square roots; on the other side on the wave function by imposing particular values for internal quantum numbers or restricting the space spanned by the wave function to a restricted subspace of the total Hilbert space.

We demonstrated in this work that one should be very careful with the approximations made to solve the eigenvalue problems numerically. Indeed, auxiliary fields, hyperspherical formalism with only one function, and Fock space reduction (magnetic or electric gluons) may cause an overestimation of the mass by amount of 100–600 MeV. However, in each case of study (magnetic or electric gluon), the less worst approximation was the only Gaussian function. For the same Hamiltonian, the single Gaussian overestimated the mass by only ~ 30 MeV where the hyperspherical formalism with only one function gave always a discrepancy around 130 MeV.

In their conclusion, the authors of [10] pointed out a numerical vector hybrid meson mass of 4.397 GeV, substantially higher than the experimental candidate $X(4260)$. The improvements of the numerical method developed in the present paper clearly favor the hybrid meson interpre-

tation of the candidate. Indeed, the real eigenvalue of the Hamiltonian (10) used in [10] is 130 MeV below the approximation with the hyperspherical formalism, the exact discrepancy with the vector $X(4260)$. We stressed, nevertheless, that the correction to the bare mass was computed with the approximative wave function and should slightly differ. It would be interesting to investigate how strong the difference is. Moreover, we have shown that the mass remains more or less stable under the approximations considered in this paper but we think that the decays properties, depending strongly on the wave functions, should change more than the mass.

It is also worth mentioning the two other vector states $Y(4325)$ from *BABAR* [24] and $Y(4360)$ from *Belle* [25]. Their mass lies close to the $X(4260)$ and mixing may cause a mass shift with respect to pure $c\bar{c}g$ hybrid states. However, the purpose of this paper was the investigation of the numerical procedure for a simple Hamiltonian for pure $c\bar{c}g$. A detail investigation of the possible mixing with other states would require a more complicated description of the system and is beyond this study.

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