

A NOTE ON OPTIMAL DEGREE-THREE SPANNERS OF THE SQUARE LATTICE

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ABSTRACT. In this short note, we prove that the degree-three dilation of the square lattice \mathbb{Z}^2 is $1 + \sqrt{2}$. This disproves a conjecture of Dumitrescu and Ghosh. We give a computer-assisted proof of a local-global property for the uncountable set of geometric graphs achieving the optimal dilation.

1. INTRODUCTION

Let P be a set of points in the Euclidean plane. A *geometric graph* G on P is an undirected graph drawn in the plane whose vertices are the points of P and whose edges are straight line segments between the corresponding points. We write $d_G(p, q)$ for the length of the shortest path between p and q that uses only edges of G ($+\infty$ if there is no such path). A geometric graph is *plane* if no two edges intersect (except possibly at a common vertex).

We measure the efficiency of a geometric graph G with a real number, called the *dilation* (or *stretch factor*, or *spanning ratio*) of G . The *dilation of a pair* (p, q) of distinct vertices of G is defined as

$$\text{dil}_G(p, q) := \frac{d_G(p, q)}{|pq|},$$

where $|pq|$ is the Euclidean distance between p and q . The *dilation* of G is the largest dilation between two vertices of G ,

$$\text{dil}(G) := \sup_{p \neq q} \text{dil}_G(p, q).$$

In other words, the dilation of G is the least $t \geq 1$ such that, for any p and q in P , the graph distance $d_G(p, q)$ is at most t times the Euclidean distance $|pq|$.

There has been extensive research on geometric graphs with low dilation which also satisfy other sparseness properties. We recall some of these results here, focussing on the following sparseness properties: being plane and having small maximum degree. We refer the reader to the survey by Bose and Smid [4] for more details and related problems.

For some constant $C > 0$, the following holds. For any finite point set P , there is a *plane* geometric graph on P with dilation at most C . The current best known constant C is due to Xia [10], who gave a rather elaborate proof that $C = 1.998$ works. The previous record was $C = 2$, a very elegant result of Chew [5]. On the other hand, it is known that C must be at least 1.4336 [8]. The

best possible constant C is conjectured to be very close to this lower bound.

The previous result still holds (for a different value of C) if we replace the condition of being plane by that of having *maximum degree 3* [6]. By considering points arranged in a grid, it is readily seen that 3 is the lowest possible maximum degree for which the result holds.

It is natural to ask whether we can simultaneously require planarity *and* small maximum degree. Define the *degree- k dilation of P* by

$$\text{dil}_k(P) := \inf_{\substack{\Delta(G) \leq k \\ G \text{ plane}}} \text{dil}(G),$$

where the infimum is taken over all plane geometric graphs on P of maximum degree k . Bose *et al.* [3] were the first to show the existence of a k (namely $k = 23$) such that the degree- k dilation of every finite point set P is bounded by an absolute constant. A lot of research has been done to determine the best possible value of k . Bonichon *et al.* [2] (and later Kanj *et al.* [9]) proved that it is possible to take $k = 4$. It is a major open problem to reduce the maximum degree down to 3.

Computing the exact value of $\text{dil}_k(P)$ for a concrete point set P is not easy in general, because the set of geometric graphs to consider is often very large. Upper bounds for the degree-3 dilation of special classes of point sets have been obtained by Biniáz *et al.* [1]. Let us mention their upper bound of $3\sqrt{2}$ for the degree-3 dilation of non-uniform rectangular grids.

The square lattice \mathbb{Z}^2 and the hexagonal lattice $\Lambda_\Delta = \mathbb{Z} \oplus e^{i\pi/3}\mathbb{Z}$ are among the few nontrivial examples of point sets for which the values $\text{dil}_k(\cdot)$ are known. The following values were obtained by Dumitrescu and Ghosh [7].

$\text{dil}_k(\Lambda)$	$k = 3$	$k = 4$	$k = 5$	$k \geq 6$
$\Lambda = \mathbb{Z}^2$	*	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
$\Lambda = \Lambda_\Delta$	$1 + \sqrt{3}$	2	2	$\frac{2}{\sqrt{3}}$

For $\text{dil}_3(\mathbb{Z}^2)$, Dumitrescu and Ghosh showed that $1 + \sqrt{2} \leq \text{dil}_3(\mathbb{Z}^2) \leq (7 + 5\sqrt{2})/\sqrt{29}$. They later gave

the improved upper bound $\text{dil}_3(\mathbb{Z}^2) \leq (3 + 2\sqrt{2})/\sqrt{5}$, which they conjectured to be tight [7]. We disprove this conjecture by giving examples of degree-3 plane geometric graphs of \mathbb{Z}^2 with dilation $1 + \sqrt{2}$.

The lower bound $\text{dil}_3(\mathbb{Z}^2) \geq 1 + \sqrt{2}$ is trivial. Indeed, let G be a geometric graph on \mathbb{Z}^2 of maximum degree 3. Let p be an arbitrary point in \mathbb{Z}^2 and let q_1, \dots, q_4 be the points in \mathbb{Z}^2 with $|pq_i| = 1$ (see Fig. 1). Since p has degree at most 3, there is some $1 \leq i \leq 4$ with $d_G(p, q_i) > 1$, and thus $\text{dil}(G) \geq \text{dil}_G(p, q_i) = d_G(p, q_i) \geq 1 + \sqrt{2}$. We will see below that there do exist graphs G which match this lower bound.

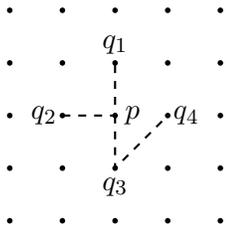


FIGURE 1. Lower bound on the degree-3 dilation of \mathbb{Z}^2 .

Definition 1.1. Let \mathcal{M} be the set of *optimal* graphs, i.e. the geometric graphs on \mathbb{Z}^2 of maximum degree 3 which have dilation $1 + \sqrt{2}$.

Definition 1.2. We also define the set \mathcal{M}_{loc} of *locally optimal* graphs: the geometric graphs G on \mathbb{Z}^2 of maximum degree 3 which satisfy the dilation constraint $\text{dil}_G(p, q) \leq 1 + \sqrt{2}$ for every pair of vertices (p, q) with $|pq| \leq \sqrt{5}$.

We claim that the geometric graphs represented in Fig. 2 are optimal. Since they are periodic, it is easy to check that they are *locally* optimal. That they indeed have dilation $1 + \sqrt{2}$ directly follows from the following result.

Theorem 1.3 (“Local-global principle”). $\mathcal{M}_{\text{loc}} = \mathcal{M}$.

The goal of this paper is to study the class of optimal graphs. Our main result is Theorem 1.3, which characterizes the optimal graphs in terms of their structure in every ball of radius $\sqrt{5}$. It will be proved in Section 2, assuming a key lemma, Lemma 2.2. We will give a computer-assisted proof of Lemma 2.2 in Section 3.

2. DEGREE-3 DILATION OF THE SQUARE LATTICE

We start this section by showing that the set of (locally) optimal graphs is very large. This seems to indicate that it might be difficult to give a fully explicit description of the set of optimal graphs, and thus makes Theorem 1.3 more interesting.

Proposition 2.1. *There are uncountably many locally optimal geometric graphs.*

Proof. For every countable sequence $(b_i)_{i \in \mathbb{N}}$ of zeroes and ones, we construct a geometric graph on \mathbb{Z}^2 as follows. Consider the periodic geometric graph in solid lines shown in Fig. 3, and let $(C_i)_{i \in \mathbb{N}}$ be an enumeration of the dashed circles. In the i -th circle, we add two vertical segments if $b_i = 1$ and two horizontal ones if $b_i = 0$.

We obtain 2^{\aleph_0} degree-3 geometric graphs in this way. Verifying that these graphs are locally optimal is a finite check by “almost periodicity”. \square

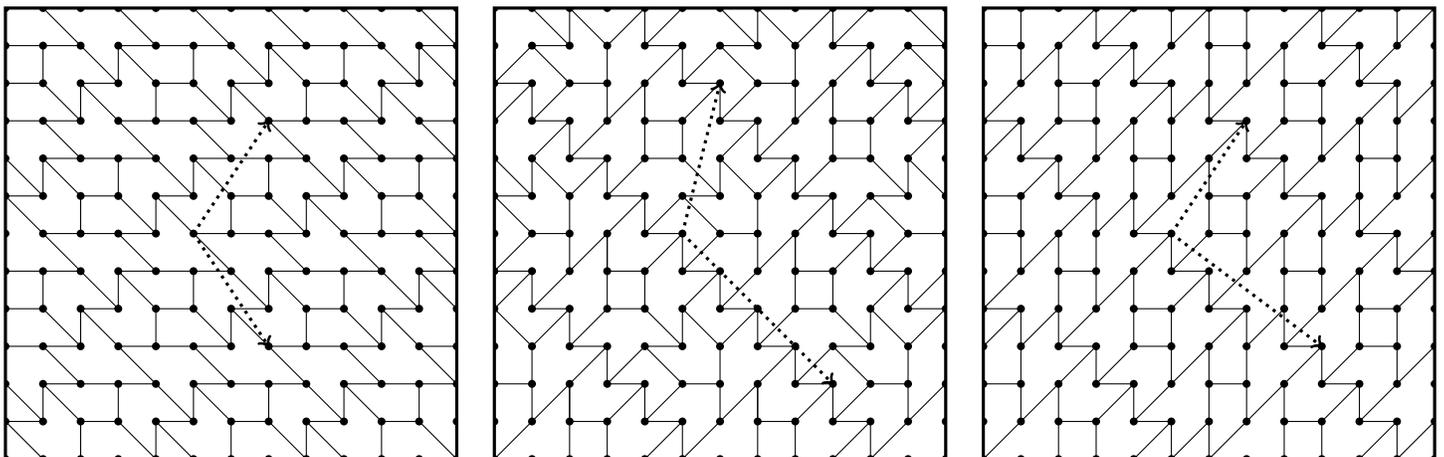


FIGURE 2. Examples of periodic degree-3 spanners of \mathbb{Z}^2 with dilation $1 + \sqrt{2}$.

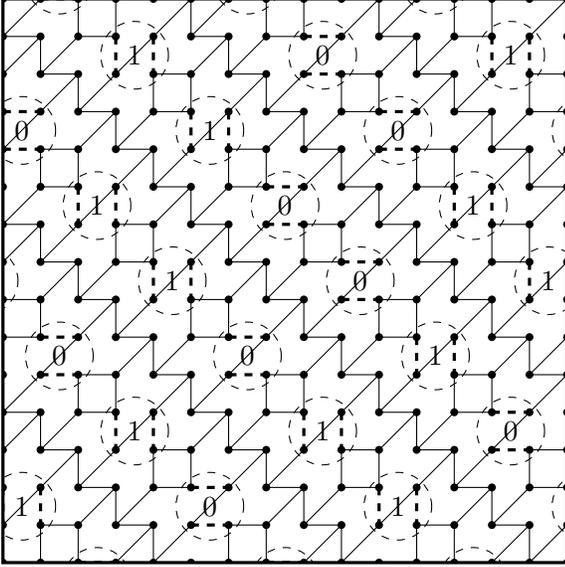


FIGURE 3. Uncountably many locally optimal geometric graphs.

Assuming Theorem 1.3, the geometric graphs we constructed in the previous proof have dilation $1 + \sqrt{2}$.

We state the following key fact, for which we will give a computer-assisted proof in Section 3.

Lemma 2.2 (“Dilation boost”). *Let $G \in \mathcal{M}_{\text{loc}}$. If $p, q \in \mathbb{Z}^2$ are such that $|pq| = \sqrt{5}$, then $d_G(p, q) \leq 3 + \sqrt{2}$.*

Remark 2.3. The definition of \mathcal{M}_{loc} only gives, a priori, that $d_G(p, q) \leq (1 + \sqrt{2})\sqrt{5} \approx 5.40$. Lemma 2.2 improves this to $d_G(p, q) \leq 3 + \sqrt{2} \approx 4.41$.

Lemma 2.4. *Define an undirected weighted (non-geometric) graph H with vertex set \mathbb{Z}^2 as follows: for $a, b \in \mathbb{Z}^2$,*

- if $|ab| \in \{1, \sqrt{2}\}$, there is an edge between a and b of weight $(1 + \sqrt{2})|ab|$;
- if $|ab| = \sqrt{5}$, there is an edge between a and b of weight $3 + \sqrt{2}$;
- otherwise, there is no edge between a and b .

Then $d_H(p, q) \leq (1 + \sqrt{2})|pq|$ for all $p, q \in \mathbb{Z}^2$.

Proof. For $p, q \in \mathbb{Z}^2$, define $N(p, q)$ to be the minimum number of “knight moves” needed to go from p to q , i.e. the least number of edges in a path from p to q consisting only of segments of Euclidean length $\sqrt{5}$. The function N is well-known; it has a lengthy but explicit formula and satisfies the inequalities

$$\frac{1}{\sqrt{5}}|pq| \leq N(p, q) \leq \frac{1}{2}|pq| + 3.$$

By using only edges of H of Euclidean length $\sqrt{5}$, we have

$$d_H(p, q) \leq (3 + \sqrt{2})N(p, q) \leq (3 + \sqrt{2}) \left(\frac{1}{2}|pq| + 3 \right).$$

Thus, the inequality $d_H(p, q) \leq (1 + \sqrt{2})|pq|$ is satisfied whenever $|pq|$ is sufficiently large ($|pq| \geq 64$). The remaining cases can easily be checked by computer, see the Python file `lemma_2_4.py`. \square

Theorem 1.3 (“Local-global principle”). $\mathcal{M}_{\text{loc}} = \mathcal{M}$.

Proof of Theorem 1.3, assuming Lemma 2.2. Let G be a locally optimal graph and let H be the weighted graph defined in Lemma 2.4. By definition of \mathcal{M}_{loc} and Lemma 2.2, we see that $d_G(p, q) \leq d_H(p, q)$ for every $p, q \in \mathbb{Z}^2$. By Lemma 2.4, we conclude that $d_G(p, q) \leq (1 + \sqrt{2})|pq|$ for all $p, q \in \mathbb{Z}^2$, i.e. $G \in \mathcal{M}$. \square

3. PROOF OF THE DILATION BOOST

This section is devoted to the proof of Lemma 2.2. We start with an easy observation: there are only short edges in a locally optimal graph.

Lemma 3.1. *The edges of every $G \in \mathcal{M}_{\text{loc}}$ are of length 1 or $\sqrt{2}$.*

Proof. Suppose that there is an edge in G of length greater than $\sqrt{2}$. Without loss of generality, this edge may be assumed to have endpoints $a = (0, 0)$ and $b = (i, j)$ for some $1 \leq i < j$.

Assume for the moment that $j > 2$. Consider the points $p = (0, 1)$ and $q = (1, 1)$. Since $G \in \mathcal{M}_{\text{loc}}$, we need to have $d_G(p, q) \leq 1 + \sqrt{2}$. As G is plane, this forces the segments pa and aq to be edges of G . However, we also need to have $d_G(a, r) \leq 1 + \sqrt{2}$, where $r = (0, -1)$. This is not possible since a already has degree 3.

If $j = 2$, i.e. $b = (1, 2)$, the same reasoning applies, exchanging the roles of a and b if necessary. \square

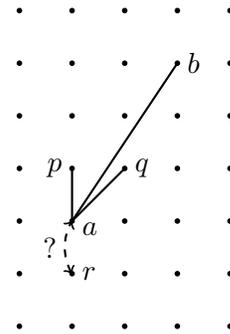


FIGURE 4. Proof of Lemma 3.1.

The previous lemma says that some “edge patterns”, namely edges of length greater than $\sqrt{2}$, cannot appear in a locally optimal graph. In Lemma 3.2, we give two more such patterns. This time, the proof is computer-assisted.

Lemma 3.2. *Let $G \in \mathcal{M}_{\text{loc}}$ and let H_1, H_2 be the edge configurations in Fig. 5. Then, neither H_1 nor H_2 (nor any translation, rotation or reflection of one of these two configurations) is a subgraph of G .*

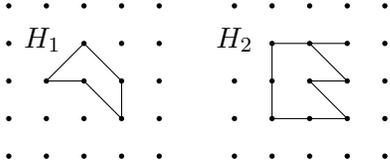


FIGURE 5. Two configurations that cannot appear in a locally optimal graph.

Proof of Lemma 3.2. Suppose that we wish to prove that a given set S of edges on \mathbb{Z}^2 is never contained in a locally optimal graph (for example, $S = H_1$ or H_2). We start with some definitions.

Definition 3.3. A *close pair* is a pair (p, q) of points in \mathbb{Z}^2 such that $|pq| \leq \sqrt{5}$.

Definition 3.4. Let (p, q) be a close pair, and let $\gamma = (v_1v_2, v_2v_3, \dots, v_{n-1}v_n)$ be a path between $v_1 = p$ and $v_n = q$ (each intermediate vertex v_i is in \mathbb{Z}^2 , the edges v_iv_{i+1} are not necessarily in S). We say that γ is an *S -admissible path between p and q* if the following conditions are verified:

- each edge v_iv_{i+1} has length 1 or $\sqrt{2}$;
- $(\mathbb{Z}^2, S \cup \gamma)$ is a plane geometric graph of maximum degree at most 3;
- the length of γ is at most $(1 + \sqrt{2})|pq|$.

For any close pair (p, q) , exactly one of the following cases must occur.

- (1) **CONTRADICTION.** There is no S -admissible path between p and q .
- (2) **SATISFACTION.** There is at least one S -admissible path between p and q which is entirely contained in S .
- (3) **DEDUCTION.** There is exactly one S -admissible path between p and q , and this path is not entirely contained in S .
- (4) **EXPLORATION.** There are several S -admissible paths between p and q , none of which is entirely contained in S .

With this terminology, we can now present the backtracking algorithm that we will use. If G is a locally optimal graph containing S , every close pair must have an S -admissible path γ consisting of edges of G . Using this fact we can, starting from S , try all possible ways of reconstructing G and hope to eventually find a contradiction in all cases. More precisely, consider Algorithm 1.

Algorithm 1 Goal: prove that a set S_0 of edges cannot be contained in a locally optimal graph

Input: S_0 , a finite set of edges

```

1: function EXPAND( $S$ )
2:   if we detect at least one close pair
     with CONTRADICTION then
3:     return
4:   else if we detect at least one close pair
     with DEDUCTION then
5:      $(p, q) \leftarrow$  any close pair with DEDUCTION
6:      $\gamma \leftarrow$  the  $S$ -admissible path between  $p$  and  $q$ 
7:     EXPAND( $S \cup \gamma$ )
8:   else
9:      $(p, q) \leftarrow$  any close pair
10:     $L \leftarrow$   $\{S$ -admissible paths between  $p$  and  $q\}$ 
11:    for  $\gamma$  in  $L$  do
12:      EXPAND( $S \cup \gamma$ )
13:    end for
14:   end if
15: end function
16:
17: EXPAND( $S_0$ ) ▷ Terminates  $\Rightarrow$  valid proof

```

Remark 3.5. On lines [2:] and [4:] of Algorithm 1, by “we detect”, it is meant that the algorithm spots a close pair with the desired property, but it might not find any even if one exists.

Claim 3.6. If Algorithm 1 terminates with input S , then S cannot be contained in a locally optimal graph.

Proof of Claim 3.6. Suppose by contradiction that Algorithm 1 terminates for a finite set of edges S_0 which is contained in a locally optimal graph G .

In the execution of Algorithm 1, there is a finite number of calls to EXPAND. Consider EXPAND(S), the last of these calls where the argument S is a (finite) set of edges of G .

Since $G \in \mathcal{M}_{\text{loc}}$, any close pair (\tilde{p}, \tilde{q}) has at least one S -admissible path $\gamma_{\tilde{p}, \tilde{q}}$ consisting of edges of G . In particular, there is no close pair with CONTRADICTION, and either lines [5-7:] or [9-13:] will be executed.

Let (p, q) be the pair chosen on line [5:] or [9:]. By construction, there will be a call to $\text{EXPAND}(S \cup \gamma_{p,q})$ on line [7:] or [12:]. However, $S \cup \gamma_{p,q}$ is a finite set of edges of G , contradicting the assumption on S . \square

Remark 3.7. Claim 3.6 holds regardless of how the points p and q are chosen on lines [5:] and [9:]. In practice, on line [9:], it is important to choose the points p and q in a way that makes the algorithm terminate in a reasonable amount of time (or terminate at all¹).

To prove Lemma 3.2, we first apply (the implemented version of) Algorithm 1 with input $S_0 = H_1$ and see that the algorithm terminates. By Claim 3.6, Lemma 3.2 is proved for H_1 .

Having just proved that H_1 cannot be contained in a locally optimal graph, we may use a slightly modified method for H_2 . We apply Algorithm 1 with input $S_0 = H_2$ after inserting the following test between lines [1:] and [2:] of Algorithm 1.

```
+1: if we detect a copy of  $H_1$  in  $S$  then
+2:   return
+3: end if
```

Again, the algorithm terminates, so Lemma 3.2 is proved for H_2 . \square

A more general version of Algorithm 1 is implemented in the Python file `proof.py`. Further explanations will be given after Algorithm 2.

For the remainder of this section, fix, by contradiction, a locally optimal graph G that violates the conclusion of Lemma 2.2: there exist $u, v \in \mathbb{Z}^2$ with $|uv| = \sqrt{5}$ that satisfy $d_G(u, v) > 3 + \sqrt{2}$. Without loss of generality, we may assume that $u = (0, 0)$ and $v = (1, 2)$.

Lemma 3.8. *Any shortest path in G between u and v must be, up to symmetry, one of the four possibilities represented in Fig. 6.*

Proof. Let P be a shortest path between u and v . By assumption, $3 + \sqrt{2} < \text{length}(P) \leq \sqrt{5}(1 + \sqrt{2})$. This leaves only a small number of possibilities for P , which are enumerated by the Python program `lemma_3_8.py`. \square

We can now adapt the algorithm given in the proof of Lemma 3.2 to give a computer-assisted proof of the dilation boost.

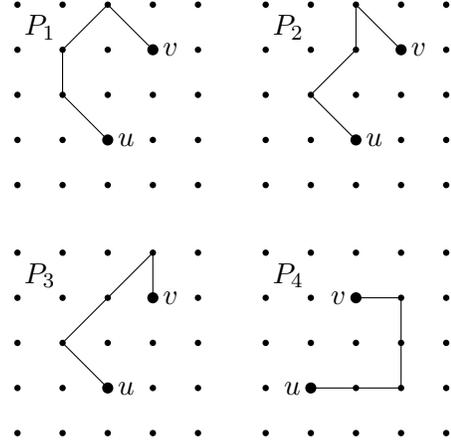


FIGURE 6. List of the possible shortest paths between u and v , up to symmetry.

Algorithm 2 Goal: prove that there is no $G \in \mathcal{M}_{\text{loc}}$ that contains a set S_0 of edges and such that a certain constraint of the form $d_G(u, v) \geq c$ is satisfied

Input: S_0 , a finite set of edges
two points $u, v \in \mathbb{Z}^2$ and a constant $c > 0$

```
1: function EXPAND( $S$ )
2:   if we detect that  $d_G(u, v) < c$ 
3:     for all  $G \in \mathcal{M}_{\text{loc}}$  containing  $S$  then
4:       return
5:   else if we detect a copy of  $H_1$  or  $H_2$  in  $S$  then
6:     return
7:   else if we detect at least one close pair
8:     with CONTRADICTION then
9:     return
10:  else if we detect at least one close pair
11:    with DEDUCTION then
12:    ( $p, q$ )  $\leftarrow$  any close pair with DEDUCTION
13:     $\gamma \leftarrow$  the  $S$ -admissible path between  $p$  and  $q$ 
14:    EXPAND( $S \cup \gamma$ )
15:  else
16:    ( $p, q$ )  $\leftarrow$  any close pair
17:     $L \leftarrow \{S\text{-admissible paths between } p \text{ and } q\}$ 
18:    for  $\gamma$  in  $L$  do
19:      EXPAND( $S \cup \gamma$ )
20:    end for
21:  end if
end function
```

21: EXPAND(S_0) ▷ Terminates \Rightarrow valid proof

¹It may be the case that, given two different ways of choosing the pairs (p, q) in Algorithm 1, the algorithm terminates for one but not for the other (with the same input S_0 in both cases). However, if it terminates for some choices of pairs (p, q) , we are certain that S_0 cannot be contained in a locally optimal graph.

Proof of Lemma 2.2. Let P be a shortest path between u and v . By Lemma 3.8, we may suppose that P is one of P_1, \dots, P_4 . Suppose $P = P_i$ for some $1 \leq i \leq 4$. Let $c_i = \text{length}(P_i)$.

We want to prove that there is no locally optimal graph G containing P_i such that $d_G(u, v) \geq c_i$. We can use the same method as in the proof of Lemma 3.2, adding the constraint $d_G(u, v) \geq c_i$ to the algorithm.

Concretely, we execute Algorithm 2 with $S_0 = P_i$, $u = (0, 0)$, $v = (1, 2)$ and $c = c_i$, and we do so for $1 \leq i \leq 4$. In each case, we observe that the algorithm terminates.

Algorithms 1 and 2 have a common implementation in the Python file `proof.py`. The file `interface.py` allows the reader to visualize the proof in real time, while `launch.py` contains the input data.

Please check `README.md` to see how to execute the different parts of the proof with customized visualization options. Implementation details and configuration instructions may also be found in the `README.md` file.

All files can be found on GitHub at <https://git.io/JTiCD> or on arXiv at <https://www.arxiv.org/src/2010.13473/anc>. \square

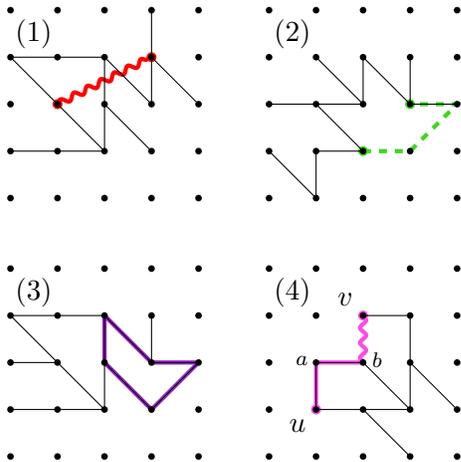


FIGURE 7. Graphical Interface examples.

We end this section by explaining the visualization produced by the `interface.py` file. Figure 7 shows an example for each type of behavior. The segments in black form the current edge set S .

- (1) CONTRADICTION: there is no S -admissible path between the two endpoints of the red string. This corresponds to line [6:] of Algorithm 2.
- (2) DEDUCTION: the path represented in green is the only S -admissible path between the two endpoints (line [8:] of Algorithm 2).
- (3) We detect a (rotated) copy of H_1 in S (line [4:]).
- (4) The condition on line [2:] of Algorithm 2 is verified with $c = 5$. Indeed, let G be a locally optimal graph containing S . There is already a path (through a) between u and b in S of length 2, and $d_G(b, v) \leq 1 + \sqrt{2}$ as $G \in \mathcal{M}_{\text{loc}}$. Thus, we must have $d_G(u, v) \leq 3 + \sqrt{2} < 5$.

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