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Symmetries of solutions for nonlinear Schrödinger equations: Numerical and Theoretical approaches

présenté par

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pour l'obtention du grade de Docteur en Sciences.

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Contents

Summary in French – Résumé en français

Cette thèse est consacrée à l'étude de l'équation de Schrödinger non-linéaire

$$-\Delta u(x) + V(x)u(x) = f(u(x)), \quad x \text{ dans } \Omega, \tag{1}$$

où $-\Delta$ est l'opérateur de Laplace, $V : \Omega \to \mathbb{R}$ est un potentiel continu, $f : \mathbb{R} \to \mathbb{R}$ est une perturbation continue non-linéaire et Ω est un domaine ouvert borné de $\mathbb{R}^N$, pour $N \geq 2$. Suivant le type de problèmes étudiés, nous travaillons avec les conditions au bord de Dirichlet ou de Neumann. Cette équation a été introduite par le physicien Erwin Schrödinger (1887–1961, Autriche, Prix Nobel en 1933) et a de nombreuses applications en astrophysique ou en mécanique quantique. Une fois l'existence de solutions établie, nous sommes intéressés par les propriétés de symétrie de celles-ci.

Tout d'abord, dans le chapitre 1, nous jouons avec le problème de Lane–Emden avec conditions au bord de Dirichlet, c'est-à-dire le problème (1) avec $f(x) = |x|^{p-2}x$, $V \equiv 0$ et nous considérons les fonctions nulles au bord du domaine. Ce problème fait référence aux astrophysiciens Jonathan Homer Lane (1819–1880, Etats-Unis) et Robert Emden (1862–1940, Suisse). Il a été introduit en 1870 dans un article qui étudiait la structure interne d'une étoile. Ici, nous considérons cette équation pour des dimensions et domaines généraux. Si $2 < p < 2^* := \frac{2N}{N-2}$ ($+\infty$ si $N = 2$), les solutions de ce problème sont les points critiques de la fonctionelle énergie $\mathscr{E}_p$ définie par

$$\mathscr{E}_p : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p.$$

En 1973 (resp. 1997), A. Ambrosetti et P. H. Rabinowitz (resp. A. Castro, J. Cossio et J. M. Neuberger) ont prouvé que ce problème possède, en plus de la solution nulle, au moins une solution de signe constant (resp. nodale) d'énergie minimale. En 1979, B. Gidas, W. N. Ni et L. Nirenberg ont montré que les solutions qui ne changent pas de signe, sur un domaine convexe, possèdent les symétries du domaine. Dans le chapitre 1, nous nous demandons si, pour des domaines généraux, les solutions nodales d'énergie minimale respectent également l'entièreté ou une partie des symétries du domaine (parité et imparité par rapport aux hyperplans de symétrie). Ces résultats sont liés aux articles [14, 39] écrits en collaboration avec D. Bonheure, V. Bouchez, C. Troestler et J. Van Schaftingen. En résumé, nous montrons que les solutions nodales d'énergie minimale :

- sur un domaine rectangulaire, pour p proche de 2, sont paires par rapport à la grande médiane et impaires par rapport à la petite ;
- sur un domaine radial, pour p proche de 2, sont paires par rapport à $N-1$ directions orthogonales et impaires par rapport à la dernière ;
- sur un domaine carré, pour p proche de 2, sont impaires par rapport au centre. De plus, nous conjecturons qu'elles sont impaires par rapport à une diagonale.

Pour de grandes valeurs de p, nous obtenons l'existence de rectangles pour lesquels les solutions nodales d'énergie minimale ne respectent pas les symétries du domaine. Nous obtenons donc une brisure de symétrie : il n'est pas possible de généraliser nos résultats de symétrie pour de grandes valeurs de p. Remarquons que notre technique retrouve les résultats de B. Gidas, W. N. Ni et L. Nirenberg pour l'étude des solutions de signe constant d'énergie minimale, au moins pour p petit. Elle donne également une méthode alternative lorsque le domaine est non-convexe.

Dans le chapitre 2, concernant les solutions nodales, nous « généralisons » le problème à des domaines non-symétriques. En fait, nous étudions la structure de la ligne nodale des solutions nodales d'énergie minimale. Il s'agit d'un travail [38] en collaboration avec C. Troestler. Nous montrons que pour des domaines convexes de dimension 2, pour p proche de 2, la ligne nodale intersecte toujours le bord du domaine. De plus, nous construisons un domaine connexe non-convexe (en fait, il est même non-simplement connexe) sur lequel, pour p

proche de 2, la ligne nodale n'intersecte pas le bord du domaine.

Les chapitres 3 à 5 sont consacrés à des problèmes plus généraux que celui de Lane–Emden. Dans le chapitre 3, dans un premier temps, nous montrons que sous l'hypothèse que $-\Delta + V$ soit défini positif, les mêmes résultats de symétrie que ceux obtenus dans le chapitre 1 sont valables. Ensuite, nous étudions l'équation de Schrödinger pour le cas du q-Laplacien

$$\begin{cases} -\Delta_q u = |u|^{p-2}u, & \text{dans } \Omega, \\ u = 0, & \text{sur } \partial\Omega, \end{cases} \tag{2}$$

où $\Delta_q u := \operatorname{div}(|\nabla u|^{q-2}\nabla u)$, pour $1 < q < p$. Ce travail [36] est une collaboration avec E. Parini. Nous obtenons que les solutions qui ne changent pas de signe (resp. nodales) d'énergie minimale convergent à un changement d'échelle près vers des solutions non-nulles du problème aux valeurs propres

$$\begin{cases} -\Delta_q u = \lambda |u|^{q-2}u, & \text{dans } \Omega, \\ u = 0, & \text{sur } \partial\Omega, \end{cases} \tag{3}$$

quand $p \to q$. Malheureusement, le manque de linéarité du q-Laplacien nous empêche d'obtenir des résultats de symétrie.

Dans le chapitre 4, nous travaillons avec des non-linéarités plus générales. En particulier, nous sommes intéressés à des non-linéarités non-homogènes. Nous donnons des hypothèses sur f de manière à ce que les résultats de symétrie donnés dans le premier chapitre restent valables. Par exemple, pour certaines valeurs de λ, les non-linéarités suivantes respecteront toutes nos hypothèses :

$$\lambda t|t|^{p-2} + (p-2)t|t|^{q-2}, \quad \lambda t(e^{t^2}-1)^{p-2} \quad \text{ou} \quad \lambda t\left(\sum_{i=1}^{k} \alpha_i |t|^{\beta_i(p-2)}\right).$$

Ce travail [12] est une collaboration avec D. Bonheure et V. Bouchez.

Dans le chapitre 5, il sera temps de travailler avec les conditions au bord de Neumann. Nous travaillons avec le problème (1) particularisé à $V \equiv 1$, $f(x) = |x|^{p-2}x$, $p > 2$ et nous considérons les fonctions à dérivée nulle sur le bord du domaine. Ce chapitre est inspiré des travaux [12, 13] écrits en collaboration avec D. Bonheure et V. Bouchez. Si $2 < p < 2^* := \frac{2N}{N-2}$ ($+\infty$ si $N = 2$), les solutions de ce problème sont les points critiques de la fonctionelle énergie $\mathscr{E}_p$

définie par

$$\mathscr{E}_p : H^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 + u^2 - \frac{1}{p}\int_\Omega |u|^p.$$

Nous montrons tout d'abord que les techniques développées dans le chapitre 1 sont toujours valables. Ceci nous permet d'étudier les symétries des solutions de signe constant et nodales d'énergie minimale lorsque p est proche de 2. En particulier, concernant les solutions de signe constant d'énergie minimale, nous obtenons directement qu'elles respectent toutes les symétries du domaine. De plus, nous améliorons ce résultat et obtenons que ces solutions doivent être égales à la constante 1 ou -1. Au contraire des conditions au bord de Dirichlet où les solutions de signe constant d'énergie minimale respectent les symétries du domaine pour tout p, on sait que ceci n'est plus nécessairement le cas pour Neumann, au moins pour de grandes dimensions. Ici, en toute dimension, nous montrons l'existence de cette brisure de symétrie : les solutions 1 et -1 ne peuvent plus être des solutions d'énergie minimale pour $p > 1 + \lambda_2$, où λ_2 est la seconde valeur propre de $-\Delta + \mathrm{id}$ avec les conditions au bord de Neumann dans $H^1(\Omega)$. Nous conjecturons d'ailleurs que $1 + \lambda_2$ est optimal.

Pour finir, rappelons que dans le chapitre 3 nous avons étudié le cas de potentiels V non-nuls tels que $-\Delta + V$ est défini positif. Si cette dernière hypothèse n'est pas vérifiée, le problème se complique. Ceci est lié au fait que la solution 0 n'est plus un minimum local de la fonctionelle énergie. Néanmoins, il est possible de montrer l'existence d'une solution non-nulle au problème. Dans ce dernier chapitre, nous développons un procédé numérique de type « mountain pass » permettant d'approcher cette solution. Nous prouvons la convergence de l'algorithme. L'implémentation nous permet de construire une conjecture sur les symétries des solutions de signe constant d'énergie minimale lorsque le potentiel V est constant et $-\Delta + V$ possède des valeurs propres négatives. Ce travail [37] est une collaboration avec C. Troestler.

Tout au long de cette thèse, nous utiliserons l'implémentation que nous avons faite des algorithmes du "mountain pass" et du "mountain pass modifié" afin d'illustrer nos résultats.

Introduction

During my schooling, I remember some professors teaching me that there exists an "universal method" to solve a scientific problem:

1. to observe (e.g. by making experiences);
2. to model (e.g. by using equations);
3. "to solve the modelization" (e.g. by approaching, finding or "studying" solutions of the equations).

In physics, many and many problems, and especially these studied during high school (just think about the use of the Newton equation, about simple mechanic problems,...), were solved following this method. In fact, many times, differential equations were used as a model (sometimes without saying it explicitely). Certainly, as we all know, it is not always so simple. Depending on the current problem, every point may be very difficult to solve: sometimes it can take lots of time or cost lots of money to make the observations and experiences, sometimes we need to simplify the problem (by making assumptions on our environment,...) to obtain a satisfying model, sometimes the solutions of the model can only be approached and not exactly obtained, and sometimes the model is just impossible to solve (or we have no idea to obtain or approach solutions). In this PhD-thesis, we deal with the third point of the "universal method" for some differential equations more or less related to physical problems. To be more precise, the differential equations studied here are mainly related to the well-known *nonlinear Schrödinger equation*

$$-\Delta u(x) + V(x)u(x) = f(u(x)), \quad x \text{ in } \Omega, \tag{4}$$

where Δ denotes the Laplacian operator, $V : \Omega \to \mathbb{R}$ is a continuous potential, $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear differentiable perturbation and Ω is an open bounded connected domain in $\mathbb{R}^N$, for $N \geq 2$. Depending on the problem, we work with Dirichlet boundary conditions (DBC; i.e. $u = 0$ on the boundary $\partial\Omega$) or Neumann boundary conditions (NBC; i.e. $\partial_\nu u = 0$ on $\partial\Omega$ where ∂_ν denotes the normal derivative). This equation is named after the theoretical physicist Erwin Schrödinger (1887–1961, Austria, Nobel Prize in 1933) and has lots of applications in astrophysics, quantum mechanics,...

Once the existence of solutions established, we are interested in properties (like symmetry properties related to the symmetries of the domain) of these solutions.

To start, we work with the so-called *Lane–Emden problem* (*LEP*) with Dirichlet boundary conditions. It is the nonlinear elliptic boundary value problem (4) with $V \equiv 0$, $f(x) = |x|^{p-2}x$ for $2 < p$, $N = 3$ and Ω a ball. It is named after the astrophysicists Jonathan Homer Lane (1819–1880, USA) and Robert Emden (1862–1940, Switzerland). This Problem (*LEP*) was introduced for the first time in 1870 by J. Lane [47]. He published the first paper studying the internal structure of a star. Physically, solutions of the Lane–Emden problem give the variation of pressure and density, for the gravitational potential of a self-gravitating, of a spherical polytropic fluid. Here, we study this equation for general dimensions ($N \geq 2$) and general domains (Ω is open bounded). If p is subcritical, i.e. $p < 2^* := \frac{2N}{N-2}$ ($2^* = +\infty$ if $N = 2$), the problem is variational. It means that the solutions of Problem (*LEP*) are the critical points of a $\mathscr{C}^1$-functional $\mathscr{E}$, i.e. functions u such that $\mathrm{d}\mathscr{E}(u) = 0$. In our case, the functional, named *energy functional*, is defined on the Sobolev space[1] $H_0^1(\Omega)$ by

$$\mathscr{E}_p(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p.$$

A priori, we just obtain weak solutions. Nevertheless, by the regularity theory (see e.g. [19]), we get classical solutions[2]. If necessary, readers can find a reference in the Master's thesis [35]: in it, existence of solutions, some symmetry results and numerical examples have already been studied in details. This work

[1] $H_0^1(\Omega)$ is the closure in $L^2(\Omega)$ of the space $\mathscr{C}_0^2(\Omega)$ for the classical norm $\left(\int_\Omega |\nabla u|^2\right)^{1/2}$.
[2] Functions in $\mathscr{C}^2(\Omega) \cap \mathscr{C}(\overline{\Omega})$

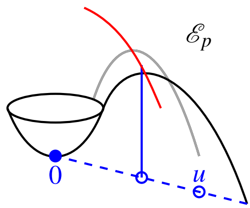


Figure 1: Energy Functional.

was the starting point of this PhD-thesis. Readers can freely download it from my web page[3].

Clearly, the zero constant function is a solution of the Lane–Emden Problem (LEP) with DBC. Let us remark that the energy $\mathscr{E}_p$ respects a Mountain-Pass type structure (see Figure 1). We mean that

- 0 is a strict local minimum of $\mathscr{E}_p$;
- for any $u \neq 0$, there exists one and only one critical point of $\mathscr{E}_p$ restricted to $\{tu\}_{t>0}$;
- for any $u \neq 0$, $\lim_{t\to+\infty} \mathscr{E}_p(tu) = -\infty$.

Concerning other solutions, in 1973, A. Ambrosetti and P. H. Rabinowitz proved that Problem (LEP) has a *ground state solution*, i.e. a non-trivial solution with minimal energy [6]. Moreover, this typical solution must be a one-signed function. To obtain this, they minimized $\mathscr{E}_p$ on

$$\left\{v \in H_0^1(\Omega) \setminus \{0\} : \left\langle \mathrm{d}\mathscr{E}_p(v), v \right\rangle = 0\right\}.$$

Concerning symmetries, in 1979, B. Gidas, W. N. Ni and L. Nirenberg [32] showed, using the elegant and now celebrated "*moving planes*" technique, that, on a convex domain, they inherit all the symmetries of the domain (e.g. evenness w.r.t. hyperplanes, see Figure 2). Moreover, on balls, a positive (resp. negative) ground state solution respects a Schwarz symmetry, i.e. u can be written as $u(x) = \tilde{u}(|x|)$ and $\tilde{u}(r)$ is nonincreasing (resp. nondecreasing). So, the symmetries of ground state solutions are already well-known at least for convex domains.

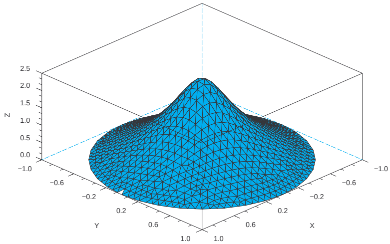


Figure 2: Ground state solution of the Lane–Emden problem on a ball.

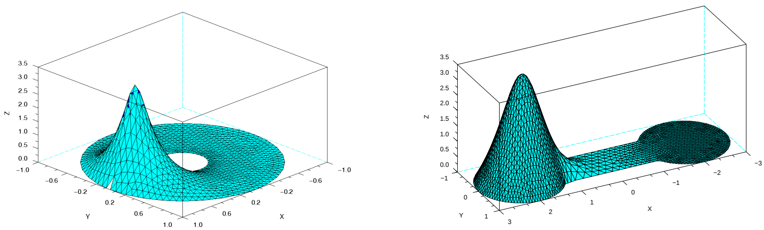
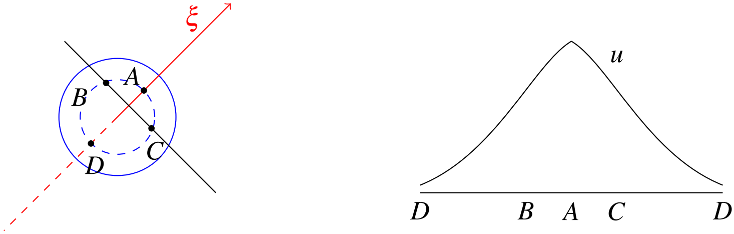


Figure 3: Ground state solution on an annulus and a dumblebell.

For a non-convex domain, the “moving planes” method does not work. In our case, on an annulus, H. Brezis and L. Nirenberg (see e.g. [20]) proved that a symmetry breaking can occur for large p (see Figure 3). We also can observe on Figure 3 a symmetry breaking on the dumblebell, a connected domain.

Turning to nodal solutions, A. Castro, J. Cossio and J. M. Neuberger [23] proved in 1997 the existence of a solution with minimal energy among all sign-changing ones, which is therefore referred to as the *least energy nodal solution (l.e.n.s.)* of Problem (*LEP*). Readers can also find reference in [68]. Since ground state solutions inherit the symmetries of the domain, at least if Ω is

[3] http://staff.umh.ac.be/Grumiau.Christopher/



$$u(A) \geqslant u(B) = u(C) \geqslant u(D)$$

Figure 4: Schwarz foliated symmetry in dimension 2.

convex, a natural question is whether least energy nodal solutions respect the symmetries of the domain Ω (e.g. oddness and evenness w.r.t. hyperplanes). Just note that the "moving planes" method cannot be used for sign-changing solutions. In fact, the story of this question is recent. Until now, the question has been only studied on radial domains. On one hand, A. Aftalion and F. Pacella [3] proved in 2004 that, on a ball, a least energy nodal solution cannot be radial (i.e. not symmetrically invariant). On the other hand, in 2005, T. Bartsch, T. Weth and M. Willem [8] obtained partial symmetry results: they showed that on a radial domain, least energy nodal solutions u have the so-called Schwarz foliated symmetry, i.e. u can be written as $u(x) = \tilde{u}(|x|, \xi \cdot x)$, where $\xi \in \mathbb{R}^N \setminus \{0\}$ and $\tilde{u}(r, \cdot)$ is nondecreasing for every $r > 0$ (see Figure 4). Such symmetry however does not imply that the zero set of the solution is an hyperplane passing through the origin as is widely believed.

The method of [8] fails when the group of the symmetries of the domain is discrete. In Chapter 1, we study the question of the symmetry of least energy nodal solutions of Problem (LEP) on more general domains. These results are related to the articles [39] written in collaboration with C. Troestler, and [14] written in collaboration with D. Bonheure, V. Bouchez and J. Van Schaftingen.

Before we mention the first symmetry result that we obtained, let us define λ_i (resp. E_i) the i^{th} eigenvalue counting without multiplicity (resp. eigenspace) of $-\Delta$ with DBC.

Theorem 1. *Assume that λ_2 is simple, i.e.* $\dim E_2 = 1$. *Then, for p close to* 2 *and any reflection R such that $R(\Omega) = \Omega$, least energy nodal solutions of Problem (LEP) have, with respect to R, the same symmetries or antisymmetries than second eigenfunctions of $-\Delta$ with DBC. Moreover, for p close to* 2*, the least energy nodal solution is unique up to symmetries of the domain and a multiplicative factor of value* -1.

In the particular case of a rectangle (see Figure 5), we deduce that the nodal line is the small median. In fact, least energy nodal solutions are odd with respect to the small median and even with respect to the orthogonal one, as functions in E_2.

When λ_2 is not simple, the situation is more delicate. Indeed, one can already figure out the difficulties on a square where the second eigenfunctions do not necessarily have an axis of symmetry whereas one would expect so for a least energy nodal solution. Nevertheless, when Ω is radial, in spite of the degeneracy of the second eigenspace, we improve the Schwarz foliated symmetry mentioned above.

Theorem 2. *Assume that Ω is a ball or an annulus. Then, for p close to* 2*, least energy nodal solutions of Problem (LEP) are radially symmetric with respect to* $N-1$ *independent directions and antisymmetric with respect to the orthogonal one. Moreover, for p close to* 2*, least energy nodal solution is unique up to rotations and multiplicative factor of value* -1.

In particular, we obtain that the nodal line is a diameter on radial domains (see Figure 6).

For general domains, we obtain a partial statement if all the second eigenfunctions enjoy some common symmetry. Just remark that, for general domains, we lose the uniqueness.

Theorem 3. *For p close to* 2*, least energy nodal solutions of Problem (LEP) respect symmetries of its projection on E_2.*

For instance, on a square, we deduce that least energy nodal solutions are antisymmetric with respect to the barycenter (see Figure 7).

In order to get further insight of symmetry properties of least energy nodal solutions, we study their asymptotic behavior as p converges to 2. We work with the classical norm $\|u\|^2 := \int_\Omega |\nabla u|^2$ in $H_0^1(\Omega)$.

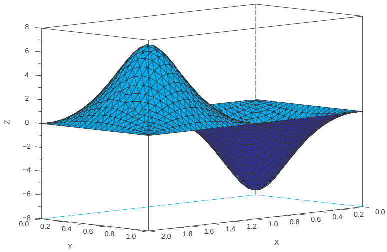


Figure 5: L.E.N.S. of the Lane–Emden problem on a rectangle.

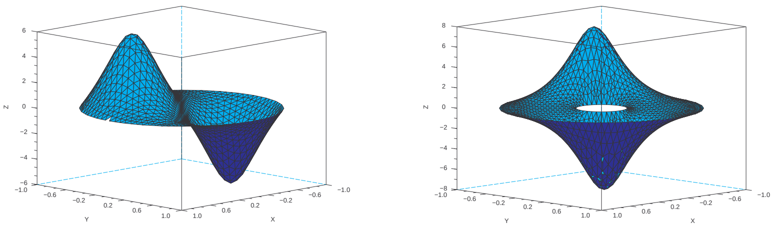


Figure 6: L.E.N.S. of the Lane–Emden problem on radial domains.

Theorem 4. *If $(u_p)_{p>2}$ are least energy nodal solutions of Problem (LEP), then there exists a real $C > 0$ such that*

$$\|u_p\| \leqslant C\lambda_2^{\frac{1}{p-2}}.$$

For any sequence $p_n \to 2$, there exists a subsequence, still denoted by p_n, such that $\lambda_2^{\frac{1}{2-p_n}} u_{p_n} \to u_* \neq 0$ in $H_0^1(\Omega)$, u_* satisfies

$$\begin{cases} -\Delta u_* = \lambda_2 u_*, & \text{in } \Omega, \\ u_* = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$\mathscr{E}_*(u_*) = \inf\{\mathscr{E}_*(u) : u \in E_2 \setminus \{0\}, \langle \mathrm{d}\mathscr{E}_*(u), u\rangle = 0\},$$

where

$$\mathscr{E}_* : E_2 \to \mathbb{R} : u \mapsto \frac{\lambda_2}{2} \int_\Omega u^2 - u^2 \log u^2.$$

Beyond its own interest, Theorem 4 leads to the following conjecture (see Section 1.4.3).

Conjecture 5. *If Ω is a square and p is close to 2, least energy nodal solutions are symmetric with respect to a diagonal and antisymmetric with respect to the orthogonal direction.*

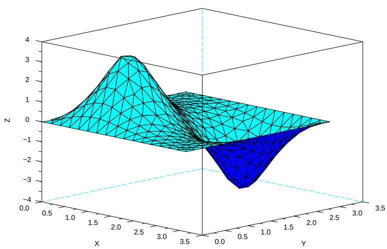


Figure 7: L.E.N.S. of the Lane–Emden problem on a square.

Theorem 4 also implies that there is no hope to improve our symmetry results for large p. By carefully exploiting the degeneracy of the square and playing with rectangles “close” to the square, we construct an example of symmetry breaking. Namely, we discuss in Section 1.5 the following result (see Figure 8).

Theorem 6. *For any $p > 2$, there exists a rectangle Ω such that any least energy nodal solutions of Problem (LEP) is neither symmetric nor antisymmetric with respect to the medians of Ω.*

Let us denote $\mathscr{N}_* := \{u \in E_2 \setminus \{0\} : \langle \mathrm{d}\mathscr{E}_*(u), u\rangle = 0\}$. Remark that a key to obtain Theorem 6 is that, on a square, an odd function with respect to a median

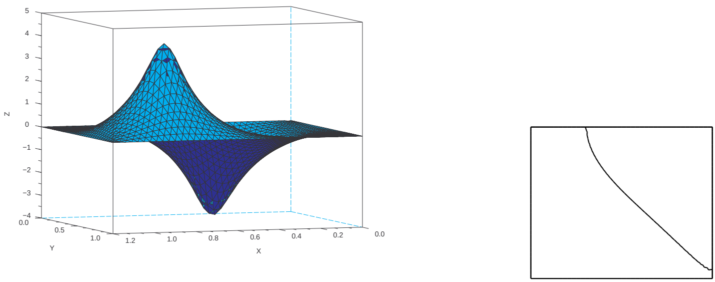


Figure 8: Non-symmetric l.e.n.s. for the Lane–Emden problem.

$u_{\text{med}} \in E_2 \cap \mathscr{N}_*$ is not a minimum of $\mathscr{E}_*$ on $\mathscr{N}_*$ (which can be numerically observed). An other way would be to directly obtain that, at p fixed, on a square, energy of least energy odd solution with respect to a diagonal is strictly less than energy of least energy odd solution with respect to a median. To do it, a potential tool is the use of assisted computer proof, as in [17, 18, 58]. This work is in progress for now, in collaboration with P. J. McKenna and C. Troestler.

Let us just mention that our techniques also work for the study of ground state solutions. Using them, we immediately obtain that, for p close to 2, ground state solutions of Problem (*LEP*) respect symmetries of its projection on E_1. So, we are able to study symmetries on non-convex domains where the "moving planes" method cannot be applied. On an annulus, we obtain in particular that, for p close to 2, ground state solutions must be radial. Let us remark that the radial symmetry of ground state solutions has already been studied for the Hénon problem[4] by D. Smets, J. Su and M. Willem in 2001 [63].

In Chapter 2, we study the structure of the nodal line, i.e. the zero set, of least energy nodal solutions for the Lane–Emden Problem (*LEP*) with DBC. In some sense, determining properties of the nodal line is a way to study the problem on non-necessarily symmetric domains. This work [38] is a collaboration with C. Troestler. In 2007, A. Aftalion and F. Pacella [3] proved that, on a radial domain, least energy nodal solutions have their nodal line intersecting $\partial\Omega$. What about general domains? In 1994, G. Alessandrini [5] (see

[4] $-\Delta u = |x|^\alpha u^p$

also [51]) proved that the nodal line of the non-zero second eigenfunctions of $-\Delta$ intersects $\partial\Omega$ at exactly two points when $\Omega \subseteq \mathbb{R}^2$ is convex. By combining Alessandrini's result with Chapter 1, we establish the following result.

Proposition 7. *For a convex domain $\Omega \subseteq \mathbb{R}^2$, not necessarily possessing any symmetry, the zero set of least energy nodal solutions intersects $\partial\Omega$, at least for p close to 2.*

Then, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili [40], in 1997, proved that there exists a connected but non-simply connected domain Ω such that $\dim E_2 = 1$ and the nodal line of the second eigenfunctions does not intersect $\partial\Omega$. This result implies the following one.

Proposition 8. *If the convexity assumption is removed, the zero set may well not touch $\partial\Omega$ anymore.*

It is a conjecture that, on a simply connected domain, the nodal line always intersects $\partial\Omega$. However, at this time, this is not proved to be true even for the linear case.

In chapter 3 to 5, we are dealing with various problems. We would like to know if asymptotic and symmetry results obtained in Chapter 1 are also working for $V \not\equiv 0$, other nonlinearities or NBC. In Chapter 3, we study perturbations of the linear part of Problem (LEP).

First, we work with a non-zero potential V:

$$\begin{cases} -\Delta u(x) + V(x)u(x) = |u(x)|^{p-2}u(x), & x \text{ in } \Omega, \\ u(x) = 0, & x \text{ on } \partial\Omega. \end{cases} \tag{5}$$

We give assumptions on V such that the results explained in Chapter 1 remain valid. We establish the following result.

Proposition 9. *If the operator $-\Delta + V$ is positive definite[5], we have*

1. *for p close to 2, a ground state solution verifies the symmetries of its projection on E_1, the first eigenspace of $-\Delta + V$;*
2. *for p close to 2, a least energy nodal solution verifies the symmetries of its projection on E_2, the second eigenspace of $-\Delta + V$.*

[5] All eigenvalues of the operator are positive.

Up to some symmetry assumptions on the eigenfunctions of $-\Delta+V$, we also obtain the existence of a symmetry breaking on some rectangles.

Second, we study the asymptotic behavior of ground state solutions and least energy nodal solutions for the q-Laplacian Lane–Emden problem with DBC:

$$\begin{cases} -\Delta_q u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\Delta_q u := \operatorname{div}(|\nabla u|^{q-2}\nabla u)$, $1<q<p<q^*$ (with $q^* = \frac{nq}{n-q}$ if $q<n$, and $q^* = +\infty$ otherwise). This work [36] is a collaboration with E. Parini. The solutions are the critical points of the energy functional $\mathscr{E}_{p,q}$ defined on the Sobolev space[6] $W_0^{1,q}(\Omega)$ and given by

$$\mathscr{E}_{p,q}(u) := \frac{1}{q}\int_\Omega |\nabla u|^q - \frac{1}{p}\int_\Omega |u|^p.$$

For $q=2$, Problem (6) corresponds to the classical Lane–Emden problem with DBC already studied. For $q\neq 2$, things seem to become more complicated, due in particular to the lack of linearity of the q-Laplacian.

Because of this, so far, we have not been able to conclude symmetries of ground state solutions and least energy nodal solutions. In fact, even for the limit problem

$$\begin{cases} -\Delta_q u = \lambda |u|^{q-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

the problem has not yet completely been solved (see Chapter 3 for details). About asymptotic behavior, it is reasonable to suppose that ground state solutions and least energy nodal solutions of Problem (6) converge when p converges to q, after being suitably scaled, to non-zero solutions of Problem (7). This result will be proved in Chapter 3.

In Chapter 4, we point out that results given in Chapter 1 also work with more general nonlinearities f_p. In particular, the technique developed in Chapter 1 does not depend on the homogeneity of the nonlinear term. In this chapter, we recall assumptions on the nonlinearity f_p such that the energy functional

[6] $W_0^{1,q}(\Omega)$ is the closure in $L^q(\Omega)$ of the space $\mathscr{C}_0^2(\Omega)$ for the classical norm $(\int_\Omega |\nabla u|^q)^{1/q}$.

given by

$$\mathscr{E}_p : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F_p(u),$$

for $F_p(t) := \int_0^t f_p(s)\,\mathrm{d}s$, is well-defined and possesses ground state and least energy nodal solutions (see [68]). Then, we give assumptions on f_p such that symmetry results and asymptotic behavior given in Chapter 1 are again applicable. For example, we study for $\lambda > 0$ the following nonlinearities f_p:

$$\lambda t|t|^{p-2} + (p-2)t|t|^{q-2}, \quad \lambda t(e^{t^2} - 1)^{p-2} \quad \text{or} \quad \lambda t\left(\sum_{i=1}^{k} \alpha_i |t|^{\beta_i(p-2)}\right).$$

This work [12] is a collaboration with D. Bonheure and V. Bouchez.

In Chapter 5, we work with the Lane–Emden problem with Neumann boundary conditions ($NLEP$). It is the nonlinear elliptic boundary value Problem (4) with $V \equiv 1$, $f(x) = |x|^{p-2}x$ for $2 < p$, and, of course, with NBC. This work is related to papers [12, 13] written in collaboration with D. Bonheure and V. Bouchez. Solutions of Problem (LEP) are critical points of the energy functional defined on the Sobolev space[7] $H^1(\Omega)$ by

$$\mathscr{E}_p(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + u^2 - \frac{1}{p}\int_\Omega |u|^p.$$

On radial domains, some methods have already been used to obtain symmetries of ground state solutions. For example, for the Hénon problem[8], direct methods have already been used to study the uniqueness and so the radial symmetry, for some p and α (see e.g. [15, 31]). Here, for general domains, we show that the symmetry results and asymptotic behavior given in Chapter 1 work (see Figure 9 for an illustration on the square). Let us just denote by λ_i (resp. E_i) the eigenvalues (resp. eigenspaces) of $-\Delta + \mathrm{id}$ in $H^1(\Omega)$ with NBC.

Theorem 10. *For p close to* 2*,*

1. *ground state solutions respect the symmetries of its projection on*[9] E_1*;*
2. *least energy nodal solutions respect symmetries of its projection on* E_2*.*

---

[7] $H^1(\Omega)$ is formed by functions in L^2 s.t. weak derivative belongs to L^2.

[8] $-\Delta u = |x|^\alpha u^p$

[9] The space E_1 is the set of constant functions.

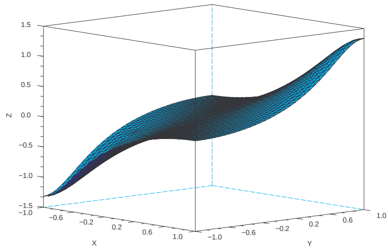


Figure 9: Ground state solution with NBC on a square.

In particular, ground state solutions must be radial functions, for p close to 2. This is an alternative to the "moving planes" method which does not apply for NBC. In fact, we even improve the result to obtain that it must be a constant function, for p close to 2, as stated in the following proposition.

Proposition 11. *For p close to 2, the only ground state solutions of Problem (NLEP) are the non-zero constant solutions.*

For large p, as in Chapter 1, we obtain the existence of a symmetry breaking for least energy nodal solutions on some rectangles. About ground state solutions, we point out differences with DBC case where ground state solutions respect the symmetries of the domain for any p (see e.g. [32]). It is already known that, on a ball, for $N \geq 3$, ground state solutions may fail to be symmetric (see e.g. [62]). In fact, for p converging to 2^*, ground state solutions converge in $H^1(\Omega)$ to a function with one peak on $\partial B(0,1)$. So, on a ball, there exists p such that ground state solutions are not radial. Here, we study for which p and domains Ω, we can expect, even in dimension $N = 2$, the first symmetry breaking.

Proposition 12. *A constant function is not a ground state solution for $p > 1 + \lambda_2$.*

To study if $1 + \lambda_2$ is optimal, we analyze as a function of the parameter p, bifurcation branches starting from the non-zero constant solutions.

Theorem 13.

- *If* $\dim E_i$ *is even (resp. odd), a sequence (resp. branch) of solutions starts from the non-zero constant solutions if and only if* $2 < p < 2^*$ *equals* $1+\lambda_i$*, for any* $i \geq 2$*;*

- *on balls, if* E_i *does not contain non-zero radial functions (as e.g. when* $i = 2$*), bifurcations are not radial;*

- *on balls, if* E_i *contains one non-zero radial function (e.g. if* $\dim E_i = 1$*), there exists at least one radial branch of solutions starting from the non-zero constant solutions.*

On radial domains, let us just remark that, inspired by the paper [3] of A. Aftalion and F. Pacella, we also obtain that non-constant ground state solutions cannot be radial.

A priori, we do not know whether the energy along bifurcations is less than the energy of the non-zero constant solution and we do not know whether one bifurcation branch emanating from $1+\lambda_2$ gives new ground state solutions or not. Nevertheless, numerical results and previous observations permit us to make the following conjecture (see Figure 10).

Conjecture 14. *In dimension* 2*, there is a non-radial symmetry breaking of ground state solutions when* p *equals* $1+\lambda_2$*.*

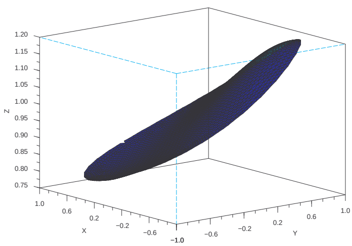


Figure 10: Ground state solution for NBC on a ball.

To conclude, recall that in Chapter 3, we studied the case where $-\Delta + V$ is positive definite. If not, zero is not a local minimum of the energy functional and so the problem does not respect a mountain pass geometry anymore. Nevertheless, P. H. Rabinowitz proved the existence of a ground state solution (see e.g. [61]). Recently, in 2009, A. Szulkin and T. Weth even obtained the existence of ground state solutions if $\Omega = \mathbb{R}^N$ [64]. Let us just remark that, now, ground state solutions could be sign-changing.

Concerning symmetry results, this time, results obtained in Chapter 1 do not work in this case. So, in the last chapter, we focus on the existence of an algorithm to approach these solutions, we prove its convergence, we implement it and we make conjectures on the symmetries, at least when V is constant. In the classical case where the energy functional respects a mountain pass structure, the mountain pass algorithm (MPA) can directly be used. This is a constrained steepest descent method. It was introduced in 1993 by Y. S. Choi and P. J. McKenna [24] and approaches solutions with Morse index[10] equals to 1 (like the ground state solutions). Typically, the algorithm is working like this.

Algorithm 15.
1. *Let $u \in H_0^1(\Omega)$ with energy functional $\mathscr{E}(u)$ strictly negative;*
2. *compute initial path: $\gamma_i = \frac{i}{N}u$, $i = 0,\dots,N$; $n \leftarrow 0$;*
3. *compute $u_n := \gamma_j = \operatorname{argmax}\{\mathscr{E}(\gamma_i) : i = 0,\dots,N\}$ and improve the localization of the maximum of "$\mathscr{E}([0,u])$" by quadratic interpolation;*
4. *compute $g_n = \nabla\mathscr{E}(u_n)$: if $\|g_n\| \leqslant \varepsilon$, then stop;*
   else deform the path: move γ_j in $\operatorname{argmin}_{s\geq 0}\mathscr{E}(u_n - sg_n)$, $n \leftarrow n+1$ and go to step 3;
5. *restart in step 2.*

In 2001, J. Zhou and Y. Li [71, 72] proved the convergence, at least up to a subsequence, of a "variant" of MPA (see Introduction of Chapter 6 for details).

For sign-changing solutions, let us mention that the modified mountain pass algorithm (MMPA) has been proposed in 1997 by J. M. Neuberger [54] (see also [25]). Based on the MPA, it allows to approach sign-changing solution but the convergence of the algorithm is not proven for now. Here, we generalize the

[10] Number of descent directions for the second derivative of the energy functional.

mountain pass algorithm to approach a non-zero solution even if $-\Delta + V$ has a negative spectrum. We prove the convergence of the algorithm.

Illustrations of the algorithm when $V = \lambda$ is a constant permit us to obtain a conjecture about symmetries of ground state solutions (see Figure 11).

Conjecture 16. *If $-\lambda_n < \lambda < -\lambda_{n-1}$, ground state solutions must respect symmetries of its projection on E_n, the n^{th} eigenspace of the Laplacian operator $-\Delta$ in $H_0^1(\Omega)$ with DBC.*

This work [37] is a collaboration with C. Troestler.

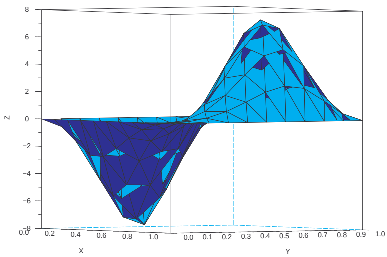


Figure 11: Ground state solution for indefinite problem.

Let us mention that throughout this thesis, we present different graphs to illustrate our results. For this, we compute MPA (resp. MMPA). While it is not sure that approximate solutions have least energy, all the other solutions that we have found numerically have a larger energy. So, unless stated otherwise, we will assume that the approximations are ground state solutions (resp. least energy nodal solutions).

To be complete, let us mention that technically, the domain Ω is triangulated with a Delaunay condition using the software Easymesh. As parameters, the distance between two nodes on the boundary of Ω is fixed at 0.05. Then, we use the Java language to compute. The algorithms relies at each step on the finite element method (see e.g. [27]). The program stops when the gradient of the energy functional at the approximations has a norm strictly less than 1.0×10^{-2} or after 2000 steps. To finish, the Scilab software is used to graph numerical solutions. Readers can find more explainations and examples in [35] and can visit the following web page[11] to get a free access code.

[11] http://staff.umh.ac.be/Grumiau.Christopher/

Chapter 1

Lane–Emden problem with DBC: symmetries of some variational solutions

As mentioned in the Introduction, we work with the so-called Lane–Emden Problem (LEP) with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{LEP}$$

where Ω is an open bounded connected domain in $\mathbb{R}^N$, $N \geq 2$ and $2 < p < 2^* := \frac{2N}{N-2}$ ($+\infty$ if $N = 2$).

The solutions are the critical points of the energy functional defined on the Sobolev space[1] $H_0^1(\Omega)$ and given by

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p.$$

We are mainly interested in the symmetries of least energy nodal solutions, i.e. sign-changing solutions with minimal energy. To study it, to avoid further renormalizations (see Section 1.1), we consider the following equivalent

[1]Closure of $\mathscr{C}_0^2(\Omega)$ for the norm $\int_\Omega |\nabla u|^2$.

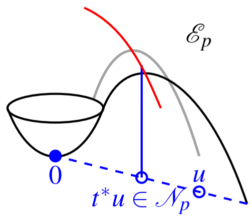


Figure 1.1: Nehari manifold.

Problem ($\mathscr{P}_p$)

$$\begin{cases} -\Delta u = \lambda_2 |u|^{p-2} u, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{P}_p$}$$

where λ_2 is the second eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ with DBC. We denote by E_2 the eigenspace related to λ_2. Weak solutions of Problem ($\mathscr{P}_p$) are critical points of the energy functional $\mathscr{E}_p$ defined on the Sobolev space $H_0^1(\Omega)$ and given by

$$\mathscr{E}_p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda_2}{p}\int_\Omega |u|^p.$$

Let us remark that, by the regularity theory (see e.g. [19]), it is possible to prove that weak solutions are in fact classical ones; they belong to $\mathscr{C}_0^2(\Omega) \cap \mathscr{C}(\overline{\Omega})$.

Clearly u is a solution (resp. least energy nodal solution) of Problem (LEP) if and only if $\lambda_2^{1/(2-p)} u$ is a solution (resp. l.e.n.s.) of Problem ($\mathscr{P}_p$). So, this rescaling does not change the symmetries of solutions.

To start the study, let us explain the structure of the energy functional $\mathscr{E}_p$. The functional $\mathscr{E}_p$ is a $\mathscr{C}^2$-functional. At $u \in H_0^1(\Omega)$, the first Frechet derivative of $\mathscr{E}_p$ at u in the direction v is given by

$$\langle \mathrm{d}\mathscr{E}_p(u), v\rangle = \int_\Omega \nabla u \nabla v - \lambda_2 \int_\Omega |u|^{p-2} uv,$$

and the second one in the directions v and w reads

$$\langle \mathrm{d}^2\mathscr{E}_p(u), v, w\rangle = \int_\Omega \nabla v \nabla w - (p-1)\lambda_2 \int_\Omega |u|^{p-2} vw.$$

Moreover, the Palais-Smale condition holds for $\mathscr{E}_p$ (see e.g. [35]), i.e. from any sequence $(u_n)_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)$, if $\mathscr{E}_p((u_n)_{n\in\mathbb{N}})$ is bounded and if the gradient

$\nabla\mathscr{E}_p(u_n)$ converges to zero then $(u_n)_{n\in\mathbb{N}}$ converges up to a subsequence. This is a classical condition generally required to obtain existence and multiplicity results. Note that the constant zero function is clearly a critical point of $\mathscr{E}_p$. Concerning non-zero critical points, seeking minima of $\mathscr{E}_p$ does not give anything since $\mathscr{E}_p$ is not bounded from below. Nevertheless, we can obtain a characterization of some non-zero critical points by defining the Nehari manifold $\mathscr{N}_p$ (see Figure 1.1) and the nodal Nehari set $\mathscr{M}_p$ by

$$\mathscr{N}_p := \{u \in H_0^1(\Omega)\setminus\{0\} : \langle \mathrm{d}\mathscr{E}_p(u), u\rangle = 0\}, \quad \mathscr{M}_p := \{u \in H_0^1(\Omega) : u^\pm \in \mathscr{N}_p\},$$

where $u^+(x) := \max(0, u(x))$ and $u^-(x) := \min(0, u(x))$.

The interest of $\mathscr{N}_p$ comes from the fact that it contains all the non-zero critical points of $\mathscr{E}_p$ and that the functions in $\mathscr{N}_p$ stay away from the zero function. This is a $\mathscr{C}^1$-manifold. A function is a ground state solution of Problem ($\mathscr{P}_p$) if and only if it minimizes $\mathscr{E}_p$ on $\mathscr{N}_p$. As the energy functional respects the Palais-Smale condition, it is possible to prove that these minima are achieved (see e.g. [6, 35]). From classical minimization arguments, we deduce that these solutions are one-signed functions: for any sign-changing solution u, $\mathscr{E}_p(u^\pm) < \mathscr{E}_p(u)$ and $u^\pm \in \mathscr{N}_p$. If $u \in H_0^1(\Omega)$ then $u \in \mathscr{N}_p$ if and only if

$$\int_\Omega |\nabla u|^2 = \lambda_2 \int_\Omega |u|^p, \tag{1.1}$$

which implies that $\left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 = \mathscr{E}_p(u)$. An other interesting fact is that for any $u \in H_0^1(\Omega)\setminus\{0\}$, there exists one and only one positive multiplicative factor $t^* > 0$ such that $t^*u \in \mathscr{N}_p$, which gives a projection from $H_0^1(\Omega)\setminus\{0\}$ to $\mathscr{N}_p$. We also have that t^*u is the unique local maximum of $\mathscr{E}_p$ in the direction u, i.e. $\mathscr{E}_p(t^*u) = \max_{t>0}(\mathscr{E}_p(tu))$.

The interest of $\mathscr{M}_p$ comes from the fact that it contains all sign-changing critical points of $\mathscr{E}_p$. A function is said to be a least energy nodal solution of Problem ($\mathscr{P}_p$) if and only if it minimizes $\mathscr{E}_p$ on $\mathscr{M}_p$. Let us just remark that, because the functions $u \mapsto u^\pm$ are not $\mathscr{C}^1$, $\mathscr{M}_p$ is not a $\mathscr{C}^1$-manifold anymore. Nevertheless, it is possible to prove that these minima are achieved (see e.g. [23, 35] or Section 3.2 for a proof for more general problems). Similar arguments as before show that these solutions have two nodal domains, just as the second eigenfunctions of $-\Delta$. If $u \in H_0^1(\Omega)$, $u^+ \neq 0$ and $u^- \neq 0$, then $u \in \mathscr{M}_p$ if and *only if*

$$\int_\Omega |\nabla u^+|^2 = \lambda_2 \int_\Omega |u^+|^p \text{ and } \int_\Omega |\nabla u^-|^2 = \lambda_2 \int_\Omega |u^-|^p. \tag{1.2}$$

We can obtain a projection from $H_0^1(\Omega)$ restricted to the sign-changing functions to $\mathscr{M}_p$: at u, we consider the maximization of the energy function $\mathscr{E}_p$ on the quarter of plane

$$\left\{\alpha u^+ + \beta u^- : \alpha \geq 0,\ \beta \geq 0\right\}.$$

In this first chapter, with the aim to obtain further symmetry results, we start by studying the asymptotic behavior of a family of least energy nodal solutions $(u_p)_{p>2}$. As $p \to 2$, in Section 1.1, our results basically show that, for general domains Ω, $(u_p)_{p>2}$ is bounded in $H_0^1(\Omega)$ and stays away from 0. This is true thanks to the rescaling that was performed to obtain the equation ($\mathscr{P}_p$). Moreover, we give a variational characterization of accumulation points of u_p.

In Section 1.2, inspired by the work of D. Smets, J. Su and M. Willem [63] which studied, for the Hénon problem (i.e. $-\Delta u = |x|^\alpha u^p$), the radiality of ground state solutions when p is close to 2, we use the boundedness to apply the following implicit function theorem and conclude symmetries in the case where $\dim E_2 = 1$ (like rectangles). Similar arguments have been also used in [28, 46],... Let us recall the implicit function theorem for the reader's convenience.

Implicit Function Theorem. *Let $\psi : A \times H \to H : (\lambda, u) \mapsto \psi(\lambda, u)$ a $\mathscr{C}^1(A \times H, H)$ function where A is in $\mathbb{R}^N$ and H is a Banach space. Assume that $\psi(\lambda_0, u_0) = 0$ and $\partial_u \psi(\lambda_0, u_0)$ is invertible.*
Then, there exists a neighbourhood $B := B(\lambda_0, r) \times B(u_0, R)$ of (λ_0, u_0) and $\gamma \in \mathscr{C}^1(B(\lambda_0, r), B(u_0, R))$ such that

$$\forall (\lambda, u) \in B, \quad \psi(\lambda, u) = 0 \text{ if and only if } u = \gamma(\lambda).$$

When $\dim E_2 \neq 1$, we cannot use directly the implicit function theorem. Nevertheless, on radial domains, as we are able to show that the degeneracy of the second eigenspace is solely due to the invariance of ($\mathscr{P}_p$) under the group rotations (see Proposition 1.3.1), we are able to use the implicit function theorem by defining a "good" subspace of $H_0^1(\Omega)$. That enables us to obtain that least energy nodal solutions are even with respect to $N-1$ independent directions *and odd with respect to the orthogonal one. Moreover, we establish uniqueness (up to the action of the group of rotations). This part is explained in Section 1.3.*

In Section 1.4, we deal with general domains. We prove that a bounded family of solutions, not necessarily ground state or least energy nodal solutions, can be distinguished by their projections on the second eigenspace: for every $M > 0$*, there exists* $\bar{p} > 2$ *such that, for every* $\alpha \in E_2 \setminus \{0\}$*, for every* $p \in (2, \bar{p})$*, Problem* ($\mathscr{P}_p$) *has at most one solution in the set* $\{u \in B(0,M) \mid P_{E_2} u = \alpha\}$*.*

This uniqueness property immediately implies partial symmetries if all the second eigenfunctions enjoy some common symmetry (like for the square where second eigenfunctions are odd with respect to the barycenter).

In Section 1.4.3, using the variational characterization of accumulation points of u_p*, we obtain a more precise symmetry result on the square. We conjecture that least energy nodal solutions are symmetric with respect to a diagonal and antisymmetric in the orthogonal direction.*

In Section 1.5, we discuss about an example of symmetry breaking: there exists a rectangle such that any least energy nodal solutions of Problem ($\mathscr{P}_p$) *is neither symmetric nor antisymmetric with respect to the medians.*

The part related to the "implicit function theorem" is inspired by the paper [39] written in collaboration with C. Troestler. The part on "general domains" is inspired by the paper [14] written in collaboration with D. Bonheure, V. Bouchez and J. Van Schaftingen. In this thesis, we will use classical tools related to the functional analysis (the Lebesgue's dominated convergence theorem, closed graph theorem, Fredholm alternative,...). Concerning the Sobolev space theory, the Sobolev's embeddings, the Rellich's embedding and the Poincaré's inequalities are mentioned in the Appendix A. Most of them have also been studied in the Master's thesis [35]. Readers can also refer to [49, 55, 70].

1.1 Asymptotic behavior

Let us fix $(u_p)_{p>2}$ a family of least energy nodal solutions of the Problem ($\mathscr{P}_p$). We prove that $(u_p)_{p>2}$ is bounded in $H_0^1(\Omega)$ and stays away from 0.

1.1.1 Upper bound

We prove it by using two different ways. The first one (see Proposition 1.1.2), is more classical but the bound is less precise than the second one (see Proposition 1.1.3). Let us fix a non-zero second eigenfunction e_2 of $-\Delta$.

Lemma 1.1.1. *For $2 < r < 2^*$, the quantities $\sup_{2<p<r} t_p^+$ and $\sup_{2<p<r} t_p^-$ are finite, where t_p^+ and t_p^- are the unique positive real numbers such that $t_p^+ e_2^+ + t_p^- e_2^- \in \mathscr{M}_p$.*

Proof. Let us fix $r \in (2, 2^*)$ and $p \in (2, r)$. Since $(t_p^\pm)^2 \|e_2^\pm\|^2 = (t_p^\pm)^p \lambda_2 \|e_2^\pm\|_p^p$, we have

$$t_p^\pm = \left(\frac{\|e_2^\pm\|^2}{\lambda_2 \|e_2^\pm\|_p^p} \right)^{\frac{1}{p-2}} > 0.$$

It is enough to show that $t_p^\pm$ converges, as $p \to 2$. We have

$$\lim_{p\to 2} \ln \left(\frac{\|e_2^\pm\|_2^2}{\lambda_2 \|e_2^\pm\|_p^p} \right)^{\frac{1}{p-2}} = \lim_{p\to 2} \frac{1}{p-2} \left(\ln\|e_2^\pm\|^2 - \ln(\lambda_2 \|e_2^\pm\|_p^p) \right).$$

As e_2 is a second eigenfunction of $-\Delta$, we have $\lambda_2 \|e_2^\pm\|_2^2 = \|e_2^\pm\|^2$. By denoting $\Omega^\pm := \{x \in \Omega : e_2^\pm \neq 0\}$, we apply the l'Hôpital's rule to obtain

$$\lim_{p\to 2} \frac{\ln(\|e_2^\pm\|^2) - \ln(\lambda_2 \|e_2^\pm\|_p^p)}{p-2} = \lim_{p\to 2} \frac{-\int_{\Omega^\pm} |e_2^\pm|^p \ln|e_2^\pm|}{\int_{\Omega^\pm} |e_2^\pm|^p} = \frac{-\int_{\Omega^\pm} |e_2^\pm|^2 \ln|e_2^\pm|}{\int_{\Omega^\pm} |e_2|^2}$$

so that

$$\lim_{p\to 2} t_p^\pm = \exp \frac{-\int_{\Omega^\pm} |e_2^\pm|^2 \ln|e_2^\pm|}{\int_{\Omega^\pm} |e_2^\pm|^2}.$$

□

Proposition 1.1.2. *The family $(u_p)_{p>2}$ is bounded in $H_0^1(\Omega)$.*

Proof. Let us define $t_p^\pm$ as in Lemma 1.1.1. As u_p belongs to the nodal Nehari set $\mathscr{M}_p$, $\|u_p\|^2 = \lambda_2 \|u_p\|_p^p$. On one hand, as the supports of e_2^+ and e_2^- are disjoint, we have that

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{p} \right) \|u_p\|^2 = \mathscr{E}_p(u_p) = \inf_{u \in \mathscr{M}_p} \mathscr{E}_p(u) &\leq \mathscr{E}_p(t_p^+ e_2^+ + t_p^- e_2^-) \\ &= \mathscr{E}_p(t_p^+ e_2^+) + \mathscr{E}_p(t_p^- e_2^-). \end{aligned}$$

On the other hand, as $t_p^+ e_2^+ \in \mathcal{N}_p$, we have

$$\mathcal{E}_p(t_p^+ e_2^+) = \left(\frac{1}{2} - \frac{1}{p}\right)(t_p^+)^2 \|e_2^+\|^2$$

and analogously for $\mathcal{E}_p(t_p^- e_2^-)$. So, we obtain

$$\|u_p\|_2^2 \leq (t_p^+)^2 \|e_2^+\|^2 + (t_p^-)^2 \|e_2^-\|^2.$$

From the uniform boundedness of t_p^+ and t_p^- (see Lemma 1.1.1), the claim follows. □

For the second approach, we obtain the upper bound by a suitable choice of test functions. In the proof, we work with the following *reduced functional*

$$\mathcal{E}_* : E_2 \to \mathbb{R} : u \mapsto \frac{\lambda_2}{2} \int_\Omega u^2 - u^2 \log u^2 \tag{1.3}$$

(where $t^2 \log t^2$ is extended continuously by 0 at $t = 0$). The critical points of $\mathcal{E}_*$ are the functions u_* such that

$$\forall v \in E_2, \quad \int_\Omega v u_* \log u_*^2 = 0. \tag{1.4}$$

Any non-trivial critical points again belong to the *reduced Nehari manifold*

$$\mathcal{N}_* := \{u \in E_2 \setminus \{0\} : \langle \mathrm{d}\mathcal{E}_*(u), u \rangle = 0\}. \tag{1.5}$$

This manifold is compact and such that $u \in \mathcal{N}_*$ if and only if

$$\int_\Omega u^2 \log u^2 = 0,$$

or equivalently if and only if

$$\mathcal{E}_*(u) = \frac{\lambda_2}{2} \|u\|_2^2.$$

Observe that, for any $u \in E_2 \setminus \{0\}$, there exists again a unique constant $t_u^* > 0$ such that $t_u^* u \in \mathcal{N}_*$.

Proposition 1.1.3. *Let $(u_p)_{p>2}$ be a family of least energy nodal solutions of Problem $(\mathscr{P}_p)$. Then we have*

$$\limsup_{p\to 2}\|u_p\|^2 = \limsup_{p\to 2}\left(\frac{\mathscr{E}_p(u_p)}{\frac{1}{2}-\frac{1}{p}}\right) \leqslant \|u_*\|^2,$$

where $u_* \in E_2$ minimizes the reduced functional $\mathscr{E}_*$ *on the* reduced Nehari manifold $\mathscr{N}_*$.

Proof. Let $w \in H_0^1(\Omega)$ be a solution of the problem

$$\begin{cases} -\Delta w - \lambda_2 w = \lambda_2 u_* \log|u_*|, & \text{in } \Omega, \\ P_{E_2} w = 0, \end{cases} \tag{1.6}$$

where P_{E_2} denotes the orthogonal projection on E_2 in $H_0^1(\Omega)$. Since u_* verifies the equation (1.4), by the Fredholm alternative, w is well-defined. Set

$$v_p := u_* + (p-2)w$$

and

$$\hat{v}_p := t_p^+ v_p^+ + t_p^- v_p^-, \quad \text{where } t_p^\pm = \left(\frac{\|v_p^\pm\|^2}{\lambda_2\|v_p^\pm\|_p^p}\right)^{\frac{1}{p-2}},$$

so that $\hat{v}_p \in \mathscr{M}_p$. We claim that $t_p^\pm \to 1$. By definition of v_p, we have

$$\begin{aligned} \|v_p^\pm\|^2 &= \int_\Omega \nabla v_p \cdot \nabla v_p^\pm \\ &= \int_\Omega (\lambda_2 u_* + \lambda_2(p-2)w + \lambda_2(p-2)u_* \log|u_*|)v_p^\pm \\ &= \lambda_2 \int_\Omega (v_p + (p-2)u_* \log|u_*|)v_p^\pm \\ &= \lambda_2 \int_\Omega |v_p|^{p-2} v_p v_p^\pm \\ &\quad + \lambda_2(p-2)\left(\int_\Omega \frac{v_p - |v_p|^{p-2}v_p}{p-2} v_p^\pm + \int_\Omega u_* \log|u_*| v_p^\pm\right). \end{aligned}$$

Since we have

$$\left|\frac{t-|t|^{p-2}t}{p-2}\right| = \frac{1}{p-2}\int_2^p \big|t\log|t|\big|\,|t|^{q-2}\,\mathrm{d}q$$
$$\leq \big|\log|t|\big|\left(|t|+|t|^{p-1}\right)$$
$$\leq \frac{1}{s}\left(|t|^{1-s}+|t|^{p-1+s}\right),$$

where $s>0$ has been choosed small enough, considering results A.1 and A.2 and applying the Lebesgue's dominated convergence theorem, we infer that

$$\lim_{p\to 2}\int_\Omega \left(\frac{v_p-|v_p|^{p-2}v_p}{p-2}+u_*\log|u_*|\right)v_p^{\pm}=0.$$

Therefore, we deduce that

$$\|v_p^{\pm}\|^2=\lambda_2\int_\Omega |v_p^{\pm}|^p+o(p-2), \tag{1.7}$$

so that $\lim_{p\to 2} t_p^{\pm}=1$. At last, since

$$\mathscr{E}_p(u_p)\leq \mathscr{E}_p(\hat{v}_p)$$

and

$$\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}\mathscr{E}_p(\hat{v}_p)=\|\nabla\hat{v}_p\|^2=\|u_*\|^2+o(1),$$

the conclusion follows easily. □

Remark 1.1.4. Geometrically, this can be pictured as follows: the natural projection of u_* on the nodal Nehari set is far from u_*, but u_* gets nearer to the Nehari manifold as $p\to 2$.

1.1.2 Lower bound

In this part, we give a lower bound for a family of least energy nodal solutions $(u_p)_{p>2}$.

Lemma 1.1.5. *For any $p\in(2,2^*)$ and $u\in H_0^1(\Omega)\setminus\{0\}$ such that $u^+\neq 0$ and $u^-\neq 0$, there exist $t^+>0$ and $t^->0$ such that $t^+u^++t^-u^-$ belongs to $\mathscr{N}_p$ and is orthogonal to e_1 in $L^2(\Omega)$, where $e_1>0$ is a first eigenfunction of $-\Delta$.*

Proof. We consider the line segment

$$T : [0,1] \to H_0^1(\Omega) \setminus \{0\} : \alpha \mapsto (1-\alpha)u^+ + \alpha u^-.$$

We project it on $\mathcal{N}_p$: for all $\alpha \in (0,1)$, there exists a unique $t_\alpha > 0$ such that $t_\alpha T(\alpha) \in \mathcal{N}_p$. For $\alpha = 0$, we have $\int_\Omega t_\alpha u^+ e_1 > 0$ and, for $\alpha = 1$, we have $\int_\Omega t_\alpha u^- e_1 < 0$. The continuity implies the existence of $\alpha^* \in (0,1)$ such that $\int_\Omega t_{\alpha^*} T(\alpha^*) e_1 = 0$ and $t_{\alpha^*} T(\alpha^*) \in \mathcal{N}_p$. We just set $t^+ := t_{\alpha^*}(1-\alpha^*)$ and $t^- := t_{\alpha^*}\alpha^*$ to conclude. □

Proposition 1.1.6. *All accumulation points of u_p as $p \to 2$ are non-zero functions.*

Proof. By Lemma 1.1.5, for all $p \in (2, 2^*)$, there exist $t_p^\pm > 0$ such that $v_p := t_p^+ u_p^+ + t_p^- u_p^-$ belongs to $\mathcal{N}_p$ and is orthogonal to e_1 in $L^2(\Omega)$.

We claim that $\|v_p\|_p \leqslant \|u_p\|_p$. As $u_p \in \mathcal{M}_p$, $u_p^+ \in \mathcal{N}_p$ maximizes the energy functional $\mathcal{E}_p$ in the direction of u_p^+ and, similarly, $u_p^- \in \mathcal{N}_p$ maximizes $\mathcal{E}_p$ in the direction of u_p^-. As the energy is the sum of the energy of the positive and negative parts, u_p maximizes the energy in the cone $K := \{t^+ u_p^+ + t^- u_p^- : t^+ > 0 \text{ and } t^- > 0\}$. Since $v_p \in \mathcal{N}_p$ implies $\lambda_2\left(\frac{1}{2} - \frac{1}{p}\right)\|v_p\|_p^p = \mathcal{E}_p(v_p)$ and given that $v_p \in \mathcal{N}_p \cap K$, we deduce

$$\lambda_2\left(\frac{1}{2} - \frac{1}{p}\right)\|v_p\|_p^p = \mathcal{E}_p(v_p) \leqslant \mathcal{E}_p(u_p) = \lambda_2\left(\frac{1}{2} - \frac{1}{p}\right)\|u_p\|_p^p.$$

Thus the claim is proved.

Let us now prove that v_p stays away from zero. By Hölder inequality, we have

$$\|v_p\|_p^2 \leqslant \|v_p\|_2^{2-2\lambda} \|v_p\|_{2^*}^{2\lambda},$$

where $\lambda := \frac{2^*}{2^*-2}\frac{p-2}{p}$. (In dimension 2, $2^* = +\infty$. In this case, we can replace 2^* by a sufficiently large q in the last inequality and use the same argument as below.) As v_p is orthogonal to e_1 in $L^2(\Omega)$, $\lambda_2 \int_\Omega v_p^2 \leqslant \|v_p\|^2$. By Sobolev's embedding theorem A.6, there exists a constant $S > 0$ such that

$$\|v_p\|_p^2 \leqslant \left(\lambda_2^{-1}\|v_p\|^2\right)^{1-\lambda}\left(S^{-1}\|v_p\|^2\right)^{\lambda} = \lambda_2^{-1}\|v_p\|^2\left(\lambda_2 S^{-1}\right)^{\lambda}.$$

As v_p belongs to $\mathcal{N}_p$, $\|v_p\|^2 = \lambda_2\|v_p\|_p^p$ and so

$$\|v_p\|_p^2 \leqslant \|v_p\|_p^p (S^{-1}\lambda_2)^{\lambda}$$

or, equivalently,

$$\|v_p\|_p \geqslant \left(S\lambda_2^{-1}\right)^{\lambda/(p-2)} = \left(S\lambda_2^{-1}\right)^{\frac{2^*}{2^*-2}\frac{1}{p}}.$$

Therefore, if u^* is the weak limit of a sequence $(u_{p_n})_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$ for some sequence $p_n \to 2$, by using Rellich's embedding theorem A.7,

$$\|u^*\|_2 = \lim_{n\to\infty} \|u_{p_n}\|_{p_n} \geqslant \liminf_{n\to\infty} \|v_{p_n}\|_{p_n} > 0.$$

□

As $\|u_p\|^2 = \lambda_2 \|u_p\|_p^p$ on $\mathscr{N}_p$, let us remark that we have that our family is also staying away from zero for H_0^1-norm.

1.1.3 Conclusion: limit equation and variational characterization

We consider a weak accumulation point u_* of a bounded family $(u_p)_{p>2}$ of solutions for Problem ($\mathscr{P}_p$). We prove that those functions verify a limit equation.

Lemma 1.1.7. *Let $(u_p)_{p>2}$ be a bounded family of solutions for Problem ($\mathscr{P}_p$). If $u_{p_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$ for some sequence $p_n \to 2$, then u_* solves*

$$\begin{cases} -\Delta u_* = \lambda_2 u_*, & \text{in } \Omega, \\ u_* = 0, & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega u_* \log|u_*| v = 0, & \forall v \in E_2. \end{cases}$$

Proof. Let $v \in H_0^1(\Omega)$. By Rellich's embedding theorem and Lebesgue's dominated convergence theorem, we deduce that

$$|u_{p_n}|^{p_n-2} u_{p_n} \to u_* \text{ in } L^2(\Omega),$$

so that

$$\begin{aligned} \int_\Omega \nabla u_* \cdot \nabla v &= \lim_{n\to\infty} \int_\Omega \nabla u_{p_n} \cdot \nabla v \\ &= \lim_{n\to\infty} \lambda_2 \int_\Omega |u_{p_n}|^{p_n-2} u_{p_n} v \\ &= \lambda_2 \int_\Omega u_* v. \end{aligned}$$

Hence, $u_* \in E_2$. To prove the last equality, taking $v \in E_2$ and multiplying the equation in Problem ($\mathscr{P}_p$) by v lead to

$$\int_\Omega (|u_{p_n}|^{p_n-2}u_{p_n} - u_{p_n})v = 0. \tag{1.8}$$

Arguing as in Proposition 1.1.3, we conclude, using Lebesgue's dominated convergence theorem and equation (1.7), that

$$\lim_{n\to\infty}\int_\Omega \frac{(|u_{p_n}|^{p_n-2}u_{p_n} - u_{p_n})v}{p_n-2} = \int_\Omega u_* \log|u_*|v.$$

Taking (1.8) into account, this completes the proof. □

Bringing together the previous lemmas, we now deduce a variational characterization for u_*.

Theorem 1.1.8. *Let $(u_p)_{p>2}$ be a family of least energy nodal solutions for Problem ($\mathscr{P}_p$). If $u_{p_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$ for some sequence $p_n \to 2$, then $u_{p_n} \to u_*$ in $H_0^1(\Omega)\setminus\{0\}$, where u_* satisfies*

$$\begin{cases} -\Delta u_* = \lambda_2 u_*, & \text{in } \Omega,\\ u_* = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$\mathscr{E}_*(u_*) = c := \inf\{\mathscr{E}_*(u) : u \in E_2\setminus\{0\}, \langle d\mathscr{E}_*(u), u\rangle = 0\} = \inf_{\mathscr{N}_*}\mathscr{E}_*.$$

Proof. By Lemma 1.1.7, $u_* \in E_2$. Observe that, as $\mathscr{E}_*(u) = \frac{1}{2}\|u\|^2$ when $u \in \mathscr{N}_*$,

$$\inf\left\{\|u\|^2 : u \in E_2\setminus\{0\},\ \int_\Omega u^2 \log u^2 = 0\right\} = 2c.$$

Applying successively Proposition 1.1.3, the weak lower semi-continuity of the norm and Lemma 1.1.7, we deduce that

$$2c = \limsup_{n\to\infty}\|u_{p_n}\|^2 \geqslant \liminf_{n\to\infty}\|u_{p_n}\|^2 \geqslant \|u_*\|^2 \geqslant 2c.$$

Hence, we conclude that $\lim_{n\to\infty}\|u_{p_n}\|^2 = \|u_*\|^2 = 2c$. This also implies immediately the strong convergence of the sequence $(u_{p_n})_{n\in\mathbb{N}}$. □

1.2 Nondegenerate case: $\dim E_2 = 1$

As we now know that a family of least energy nodal solutions $(u_p)_{p>2}$ for Problem ($\mathscr{P}_p$) is bounded in $H_0^1(\Omega)$ and stays away from the zero function, we now are able to deduce some symmetries of u_p, at least in the case where $\dim E_2 = 1$ and for p close to 2. The main idea is to apply the implicit function theorem (see page 34). Let us fix $e_2 \in E_2$ such that $\|e_2\| = 1$.

1.2.1 Method based on the implicit function theorem

Lemma 1.2.1. *In dimension $N \geqslant 2$, in $H_0^1(\Omega) \times \mathbb{R}$, Problem* (1.9)

$$\begin{cases} -\Delta u = \lambda |u|^{p-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \|u\| = 1, \end{cases} \tag{1.9}$$

possesses a single curve of solutions $p \mapsto (p, \tilde{u_p}, \lambda_p^*)$ defined for $2 < p$ close to 2 and starting from $(2, e_2, \lambda_2)$. It also possesses a single curve of solutions starting from $(2, -e_2, \lambda_2)$ which is given by $p \mapsto (p, -\tilde{u_p}, \lambda_p^*)$.

Proof. Let us define

$$\begin{aligned} \psi : (2, 2^*) \times H_0^1(\Omega) \times \mathbb{R} &\to H_0^1(\Omega) \times \mathbb{R} \\ (p, u, \lambda) &\mapsto \big(u - \lambda(-\Delta)^{-1}(|u|^{p-2}u), \|u\|^2 - 1\big). \end{aligned}$$

The first component is the H_0^1-gradient of the following energy functional

$$\mathscr{E}_{p,\lambda} : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\|u\|^2 - \frac{\lambda}{p}\|u\|_p^p.$$

The existence and local uniqueness of a branch emanating from $(2, e_2, \lambda_2)$ follows from the implicit function theorem and the closed graph theorem if we prove that the derivative of $\mathscr{E}$ with respect to (u, λ) at the point $(2, e_2, \lambda_2)$ is bijective on $H_0^1(\Omega) \times \mathbb{R}$. We have,

$$\begin{aligned} &\partial_{(u,\lambda)} \psi(2, e_2, \lambda_2)(v, t) \\ &\qquad = \Big(v - \lambda_2(-\Delta)^{-1} v - t(-\Delta)^{-1} e_2,\ 2\int_\Omega \nabla e_2 \nabla v\Big). \end{aligned} \tag{1.10}$$

For the injectivity, let us start by showing that $\partial_{(u,\lambda)}\psi(2,e_2,\lambda_2)(v,t) = 0$ if and only if

$$\begin{cases} v - \lambda_2(-\Delta)^{-1}v = 0, \\ t = 0, \\ v \text{ is orthogonal to } e_2 \text{ in } H_0^1(\Omega). \end{cases} \tag{1.11}$$

It is clear that (1.11) is sufficient. For its necessity, remark that the nullity of second component of (1.10) implies that e_2 is orthogonal to v in $H_0^1(\Omega)$ and thus also in $L^2(\Omega)$ because e_2 is an eigenfunction. Taking the L^2-inner product of the first component of (1.10) with e_2 yields $t = 0$, hence the equivalence is complete. Now, the only solution of (1.11) is $(v,t) = (0,0)$ because the first equation and the dimension 1 of E_2 imply that $v = \alpha e_2$, for some $\alpha \in \mathbb{R}$. The third property implies $v = 0$. This concludes the proof of the injectivity of $\partial_{(u,\lambda)}\psi(2,e_2,\lambda_2)$.

Let us now show that, for any $(w,s) \in H_0^1(\Omega)\times\mathbb{R}$, the equation

$$\partial_{(u,\lambda)}\psi(2,e_2,\lambda_2)(v,t) = (w,s)$$

always possesses at least one solution $(v,t) \in H_0^1(\Omega)\times\mathbb{R}$. One can write $w = \bar{w}e_2 + \widetilde{w}$ for some $\bar{w} \in \mathbb{R}$ and $\widetilde{w}$ orthogonal to e_2 in $H_0^1(\Omega)$. Similarly, one can decompose $v = \bar{v}e_2 + \widetilde{v}$. Arguing as for the first part, the equation can be written

$$\begin{cases} \widetilde{v} - \lambda_2(-\Delta)^{-1}\widetilde{v} = \widetilde{w}, \\ t = \lambda_2\bar{w}, \\ \bar{v} = s/2. \end{cases} \tag{1.12}$$

By the principle of symmetric criticality, the solution $\widetilde{v}$ is the minimizer of the functional

$$E_2^{\perp} \to \mathbb{R} : \widetilde{v} \mapsto \int_\Omega |\nabla\widetilde{v}|^2 - \lambda_2|\widetilde{v}|^2 - \int_\Omega \nabla\widetilde{w}\nabla\widetilde{v}.$$

This concludes the proof that $\partial_{(u,\lambda)}\psi(2,e_2,\lambda_2)$ is onto and thus of the existence and uniqueness of the branch emanating from $(2,e_2,\lambda_2)$.

It is clear that $p \mapsto (p,-u_p^*,\lambda_p^*)$ is a branch emanating from $(2,-e_2,\lambda_2)$ and, using as above the implicit function theorem at that point, we know it is the only one. □

We deduce now some symmetries when $\dim E_2 = 1$.

Theorem 1.2.2. *Assume that λ_2 is simple. Then, for p close to 2 and any reflection R such that $R(\Omega) = \Omega$, least energy nodal solutions of Problem ($\mathscr{P}_p$) respect the symmetries or antisymmetries of e_2 with respect to R. Moreover, for p close to 2, least energy nodal solution of Problem ($\mathscr{P}_p$) is unique up to symmetries of the domain and multiplicative factor of value -1.*

Proof. Let $(u_p)_{p>2}$ be a family of solutions of Problem ($\mathscr{P}_p$). On one hand, for any sequence $p_n \to 2$, there exists a subsequence, still denoted p_n, such that u_{p_n} weakly converges in $H_0^1(\Omega)$ to some $u^* = \alpha e_2 \in E_2 \setminus \{0\}$ (thanks to 1.1.8).

On the other hand, notice that u_p is a solution of Problem ($\mathscr{P}_p$) if and only if $\big(u_p/\|u_p\|, \lambda_2 \|u_p\|^{p-2}\big)$ is a solution of (1.9). Because $(u_{p_n})_{n\in\mathbb{N}}$ stays bounded away from 0, one has

$$\left(\frac{u_{p_n}}{\|u_{p_n}\|}, \lambda_2 \|u_{p_n}\|^{p_n-2}\right) \underset{n}{\to} \big(\operatorname{sign}(\alpha) e_2, \lambda_2\big).$$

Then, when p_n is close enough to 2, Lemma 1.2.1 implies that

$$\frac{u_{p_n}}{\|u_{p_n}\|} = \operatorname{sign}(\alpha)\, u^*_{p_n}.$$

Hence, this claimed uniqueness of u_p up to its sign. Let us define the reflection R such that $R(\Omega) = \Omega$. We prove that u_{p_n} respects symmetries of e_2 with respect to R. W.l.o.g., assume that e_2 is odd (resp. even) with respect to R (as $\dim E_2 = 1$, we are sure that e_2 is odd or even with respect to R). To show the oddness (resp. evenness) of u_{p_n}, let us consider u'_{p_n} the anti-symmetric (resp. symmetric) of u_{p_n} with respect to R defined by $u'_{p_n} := \mp u_{p_n}(x - 2(x.R)R)$ where $x.R$ denotes the inner product in $\mathbb{R}^N$. Because e_2 is odd (resp. even) with respect to R, $u'_{p_n} \to \alpha e_2$ with the same α as for u^*. Arguing as before, we conclude that

$$\frac{u_{p_n}}{\|u_{p_n}\|} = \operatorname{sign}(\alpha)\, u^*_{p_n} = \frac{u'_{p_n}}{\|u'_{p_n}\|}$$

and therefore that u_{p_n} is odd (resp. even) with respect to R. □

1.2.2 Example: the rectangle

As an example, we immediately have that, on rectangles, least energy nodal solutions are odd with respect to the small median and even with respect to the large one, for p close to 2. Let us illustrate this numerically. Consider a rectangle of sidelengths 1 and 2. Figure 1.2 shows a nodal solution u of the problem $-\Delta u = u^3$ with Dirichlet boundary conditions obtained by MMPA and level lines at levels -1, -0.5, 0, 0.5 and 1.

MPA suggests that u^+ and u^- are ground state solutions of Problem ($\mathscr{P}_p$) on the squares defined by the nodal regions. One sees that u is antisymmetric with respect to the shortest median, which is thus also the nodal line.

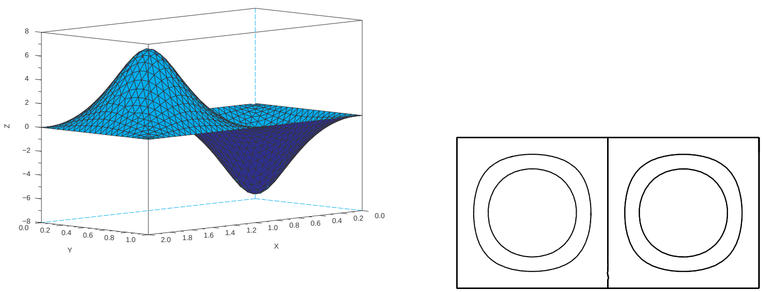


Figure 1.2: MMPA solution of the Lane–Emden problem on a rectangle.

Initial function	$\min u$	$\max u$	$\mathscr{E}_4(u^+)$	$\mathscr{E}_4(u^-)$
$\cos(\pi x)\cos(\pi y)\,x\,(x-2)\,y\,(y-1)$	-6.6	6.6	37.2	37.2

Table 1.1: Characteristics of a l.e.n.s. on a rectangle.

1.3 Radial domains

Now, we study the case where the domain is radial. Let us mention that, as $\dim E_2 > 1$, $\partial_{(u,\lambda)}\psi(2, e_2, \lambda_2)$ is not bijective. To overcome this problem, we study the structure of E_2.

1.3.1 Study of E_2

In this section, we use the interlacing properties of zeros of (cross-products of) Bessel functions and results by H. Kalf on the symmetries of spherical harmonics [44] to show that all eigenfunctions of eigenvalue λ_2 have an hyperplane as nodal set with respect to which the function is odd. These results follow from well-known formulae for dimensions 2 and 3 (see e.g. [57]) but we could not find a ready-to-use reference for higher dimensions. More precisely, we discuss the following proposition. Let us denote by S^N the unit sphere of $\mathbb{R}^N$.

Proposition 1.3.1. *Let $\Omega \subseteq \mathbb{R}^N$, $N \geqslant 2$ be a ball or an annulus and let $\xi \in S^N$ be a direction. The subspace of eigenfunctions for the second eigenvalue λ_2 of $-\Delta$ on Ω with Dirichlet boundary conditions which are invariant under rotations around ξ has dimension* 1. *Moreover, these eigenfunctions are odd in the direction ξ. For any second eigenfunctions e_2, there exists $\xi \in S^N$ such that e_2 is invariant under rotations around ξ.*

Eigenfunctions $u : \Omega \to \mathbb{R}$ of $-\Delta$ are the solutions of

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

In (hyper)spherical coordinates $x = r\theta$ with $r \in (0, +\infty)$ and $\theta \in S^N$, the equation $-\Delta u = \lambda u$ reads (see e.g. [45, 53], or reprove it using a local orthogonal parametrisation of S^N):

$$\partial_r^2 u + \frac{N-1}{r}\partial_r u - \frac{1}{r^2}\left(-\Delta_{S^N} u\right) = -\lambda u,$$

where Δ_{S^N} denotes the Laplace-Beltrami operator on the unit sphere S^N. By the method of separation of variables, we search functions $u(r,\theta) = R(r)S(\theta)$ satisfying

$$\begin{cases} \partial_r^2 R + \dfrac{N-1}{r}\partial_r R + \left(\lambda - \dfrac{\mu}{r^2}\right) R = 0, \\ -\Delta_{S^N} S = \mu S. \end{cases} \tag{1.13}$$

The eigenvalues μ_k of the Laplace-Beltrami operator $-\Delta_{S^N}$ are well-known (see e.g. [45, 67]):

$$\mu_k = k(k+N-2), \quad \text{for } k \in \mathbb{N}.$$

The corresponding eigenfunctions S_k are called spherical harmonics. These are restrictions to the unit sphere ($S = P\restriction_{S^N}$) of homogeneous polynomials $P : \mathbb{R}^N \to \mathbb{R}$ satisfying $\Delta P = 0$ in $\mathbb{R}^N$. The eigenfunctions of eigenvalue μ_k are the restrictions of the homogeneous polynomials of degree k among those [53, p. 39].

In order for $R(r) =: r^{-\frac{N-2}{2}} B(\sqrt{\lambda}\, r)$ to be solution of the first equation of the system (1.13) with $\mu = \mu_k$, it is necessary and sufficient that the function $s \mapsto B(s)$ satisfies

$$\partial_s^2 B + \frac{1}{s}\partial_s B + \left(1 - \frac{\nu^2}{s^2}\right) B = 0, \tag{1.14}$$

where $\nu^2 := \mu_k + \frac{(N-2)^2}{4} = \left(k + \frac{N-2}{2}\right)^2$. Solutions of equation (1.14) are linear combinations of the Bessel functions of the first kind

$$J_\nu(s) := \left(\frac{s}{2}\right)^\nu \sum_{k=0}^{+\infty} \frac{(-1)^k s^{2k}}{2^{2k}\Gamma(k+1)\Gamma(\nu+k+1)}$$

where, for any $c > 0$, $\Gamma(c) := \int_0^{+\infty} x^{c-1} e^{-x}\, \mathrm{d}x$ denotes the gamma functions, and of the second kind

$$Y_\nu(s) := \lim_{z \to \nu} \frac{J_z(s)\cos(z\pi) - J_{-z}(s)}{\sin(z\pi)}.$$

It is easy to prove by induction that $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$.

Therefore, the solutions of the first equation of (1.13) with $\mu = \mu_k$ are

$$R(r) = r^{-\frac{N-2}{2}} \Big(aJ_\nu\big(\sqrt{\lambda}\, r\big) + bY_\nu\big(\sqrt{\lambda}\, r\big)\Big), \qquad a, b \in \mathbb{R}, \tag{1.15}$$

where $\nu = k + \frac{N-2}{2}$.

Let us now distinguish two cases.

If Ω is a ball—which can be assumed to be of radius one without loss of generality—, the function Y_ν cannot appear in (1.15) because $\lim_{r\to 0} Y_\nu(r) = -\infty$ and image at 0 must be finite. Imposing the Dirichlet boundary conditions, we obtain that the eigenvalue λ must be the square of a positive root of J_ν. If the radius of Ω goes to $+\infty$, let us remark that eigenvalues go to 0. As is customary, let us denote $0 < j_{\nu,1} < j_{\nu,2} < \cdots$ the infinitely many positive roots of J_ν. The

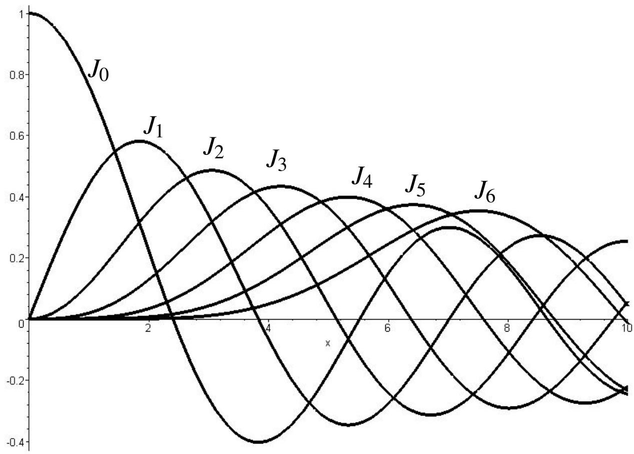


Figure 1.3: Bessel functions.

interlacing property of the roots (see e.g. M. Abramowitz and A. Segun [1, § 9.5.2, p. 370]) says,

$$\forall \nu \geqslant 0, \qquad j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < \cdots \tag{1.16}$$

The figure 1.3 illustrates this fact for the first few values of ν. So, in particular, we obtain that $j^2_{\frac{N-2}{2},1}$ is the first eigenvalue of $-\Delta$ and $j^2_{\frac{N}{2},1}$ its second. Therefore, the eigenfunctions for the second eigenvalue of $-\Delta$ are given by

$$r^{-\frac{N-2}{2}} J_{N/2}\big(\sqrt{\lambda}\, r\big) S(\theta) \quad \text{with } \lambda = j^2_{\frac{N}{2},1},$$

where S is a spherical harmonic of eigenvalue μ_1.

To conclude it suffices to use the fact that, for all directions $\xi \in S^N$ and $k \in \mathbb{N}$, there exists exactly one (apart from a multiplicative constant) homogeneous polynomial P of degree k such that $\Delta P = 0$ and is invariant under rotations around ξ (see e.g. [44, 53]). Therefore, there exists one and only one spherical harmonic of eigenvalue μ_1 that is invariant under rotations around a given direction ξ. Moreover, this spherical harmonic is the restriction to the sphere of an homogeneous polynomial of degree 1 — i.e. a linear functional — and is consequently odd in the direction ξ.

Now let us turn to the second case where Ω is an annulus. Without loss of generality, one can assume that its internal radius is 1 and its external radius is $\rho \in (1,+\infty)$. Imposing the Dirichlet boundary conditions on (1.15) leads to the system

$$\begin{cases} aJ_\nu(\sqrt{\lambda}) + bY_\nu(\sqrt{\lambda}) = 0, \\ aJ_\nu(\sqrt{\lambda}\rho) + bY_\nu(\sqrt{\lambda}\rho) = 0. \end{cases}$$

A non-trivial solution (a,b) of this system exists if and only if

$$\sqrt{\lambda} \text{ is a root of the function } s \mapsto J_\nu(s)Y_\nu(s\rho) - Y_\nu(s)J_\nu(s\rho).$$

It is known that this function possesses infinitely many positive zeros that we will note $0 < \chi_{\nu,1} < \chi_{\nu,2} < \cdots$ Again an interlacing theorem for these zeros holds [16, p. 1736]: for all $\nu \geqslant 0$, $\chi_{\nu,1} < \chi_{\nu+1,1} < \chi_{\nu,2} < \chi_{\nu+1,2} < \cdots$ As before, we deduce that the first eigenvalue happens for $k = 0$ (constant spherical harmonic) and $\nu = (N-2)/2$, while the second is when $k = 1$ and $\nu = N/2$. We then conclude in the same way as for the ball.

1.3.2 Method based on the "Radial IFT"

To be able to apply the implicit function theorem (see page 34), let us fix a direction $\xi \in S^N$. In the Introduction, we mentioned that, in 2005, T. Bartsch, T. Weth and M. Willem [8] proved that, on radial domains, least energy nodal solutions respect a Schwarz foliated symmetry. So, they are rotationally invariant around a direction. Without loss of generality, up to rotations, we can assume that least energy nodal solutions are rotationally invariant around ξ. In Section 1.3.1, we obtained that the dimension of the space $E_2 \cap \mathrm{Fix}(G)$, where $\mathrm{Fix}(G)$ is the space $H_0^1(\Omega)$ restricted to the functions rotationally invariant around ξ, equals 1. Moreover, these functions are odd in direction ξ. If we denote by e_2 a function in $E_2 \cap \mathrm{Fix}(G)$, we obtain the following result.

Proposition 1.3.2. *All weak accumulation points of the family $(u_p)_{p>2}$ as $p \to 2$ are invariant under rotations leaving ξ fixed and have the form αe_2 for some $\alpha \in \mathbb{R} \setminus \{0\}$.*

By working exactly in the same way as in Section 1.2.1 but by using the implicit function theorem (see page 34) on $\mathrm{Fix}(G)$ (instead of all $H_0^1(\Omega)$), we directly obtain the following result.

Theorem 1.3.3. *Assume that Ω is a ball or an annulus. Then, for p close to 2, least energy nodal solutions of Problem ($\mathscr{P}_p$) are radially symmetric with respect to $N-1$ independent directions and antisymmetric with respect to the orthogonal one. Moreover, for p close to 2, least energy nodal solution is unique up to rotations and multiplicative factor of value -1.*

1.3.3 Examples: the ball and the annulus

Let us illustrate this on the unit ball in $\mathbb{R}^2$. Figure 1.4 depicts a nodal solution u of the problem $-\Delta u = u^3$ with Dirichlet boundary conditions. As previously, the application of the MPA to u^+ and u^- suggest that the algorithm indeed caught a least energy nodal solution. One should note that, as shown by A. Aftalion and F. Pacella [3], u is not radial. As shown by T. Bartsch, T. Weth and M. Willem [8], it is symmetric with respect to a direction. Moreover, it is antisymmetric with respect to the orthogonal one.

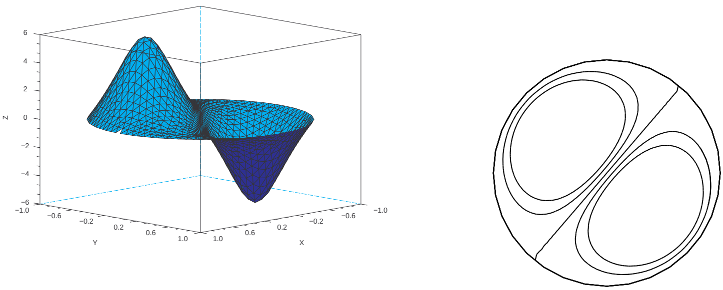


Figure 1.4: MMPA solution of the Lane–Emden Problem on a ball.

	Initial function	$\min u$	$\max u$	$\mathscr{E}(u^+)$	$\mathscr{E}(u^-)$
B	$\cos(\pi r/2)$	-5.8	5.8	29.7	29.7
A	$\cos(\pi r/2)\cos(2\pi r)\cos(\pi r)$	-7.1	7.1	45.4	45.4

Table 1.2: Characteristics of l.e.n.s. on radial domains.

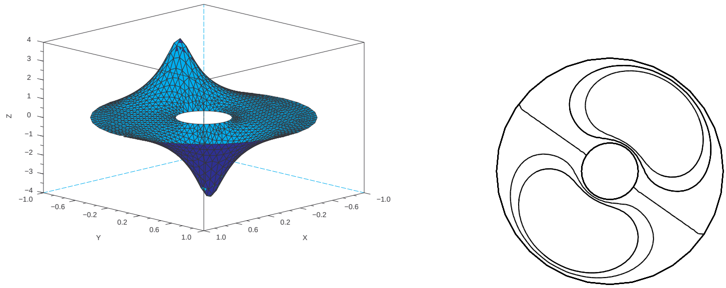


Figure 1.5: MMPA solution of the Lane–Emden Problem on an annulus.

Figure 1.5 depicts the same thing for the annulus $B(0,1)\setminus B(0,0.25)$ in $\mathbb{R}^2$. In Table 1.2, we have the characteristics of the solutions. We used r to denote $\sqrt{x^2+y^2}$.

1.4 General domains

For general domains, the implicit function theorem (see page 34) cannot be used in the same way as before. This time, we do not know partial symmetry. So, we define an other way to work.

1.4.1 A uniqueness result

Bounded families of solutions for Problem ($\mathscr{P}_p$) can be distinguished by their projections on the second eigenspace E_2. This result will follow from Proposition 1.4.2 below.

Lemma 1.4.1. *Let $N \geqslant 3$. There exists $\varepsilon > 0$ such that if $\|a(x)-\lambda_2\|_{N/2} < \varepsilon$ and u solves the boundary value problem*

$$\begin{cases} -\Delta u = a(x)u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

then either $u = 0$ or $P_{E_2}u \neq 0$.

Proof. Assume by contradiction that there exists a nontrivial solution u such that $P_{E_2}u = 0$. Let $w = P_{E_1}u$ and $z = P_{(E_1+E_2)^\perp}u$. Taking successively w and z as test functions and using Poincaré's inequalities A.3 and Sobolev's inequalities, we infer that there exists a positive constant C such that

$$\begin{aligned}
\|w\|^2 &= \lambda_2 \int_\Omega w^2 + \int_\Omega (a(x) - \lambda_2)uw \\
&\geqslant \frac{\lambda_2}{\lambda_1}\|w\|^2 - C\|a(x) - \lambda_2\|_{\frac{N}{2}}\|w\|\|u\|, \\
\|z\|^2 &= \lambda_2 \int_\Omega z^2 + \int_\Omega (a(x) - \lambda_2)uz \\
&\leqslant \frac{\lambda_2}{\lambda_3}\|z\|^2 + C\|a(x) - \lambda_2\|_{\frac{N}{2}}\|z\|\|u\|.
\end{aligned}$$

We deduce that

$$\begin{aligned}
\|w\| &\leqslant \frac{\lambda_1 C}{\lambda_2 - \lambda_1}\|a(x) - \lambda_2\|_{\frac{N}{2}}\|u\|, \\
\|z\| &\leqslant \frac{\lambda_3 C}{\lambda_3 - \lambda_2}\|a(x) - \lambda_2\|_{\frac{N}{2}}\|u\|.
\end{aligned}$$

Since $P_{E_2}u = 0$, we now conclude that there exists a positive constant C such that

$$\|u\|^2 = \|w\|^2 + \|z\|^2 \leqslant C\|a(x) - \lambda_2\|^2_{\frac{N}{2}}\|u\|^2.$$

When $\|a(x) - \lambda_2\|_{\frac{N}{2}}$ is small enough, this leads to a contradiction, so that the conclusion follows. □

Observe that, if $N = 2$, the same statement can be formulated with the $L^{\frac{N}{2}}$-norm replaced by any L^q-norm with $1 < q < +\infty$. We only need to replace in the proof the use of the Sobolev inequality by the embedding in any $L^p(\Omega)$ with $1 < p < +\infty$.

Using the previous lemma, we deduce the following proposition.

Proposition 1.4.2. *For every $M > 0$, there exists $\bar{p} > 2$ such that, for every $p \in (2, \bar{p})$, if $u_p, v_p \in \{u \in B(0,M) : P_{E_2}(u) \notin B(0, \frac{1}{M})\}$ solve the boundary value problem*

$$\begin{cases} -\Delta u = \lambda_2 |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

then $P_{E_2}u_p = P_{E_2}v_p$ implies $u_p = v_p$.

Proof. It is enough to consider $p_n \to 2$, $(u_n)_{n\in\mathbb{N}} \subseteq B(0,M)$ and $(v_n)_{n\in\mathbb{N}} \subseteq B(0,M)$ be two sequences such that, for any $n \in \mathbb{N}$, u_n and v_n solve Problem $(\mathscr{P}_p)$ with $p = p_n$ and $P_{E_2}u_n = P_{E_2}v_n \notin B(0,\frac{1}{M})$. We prove that $u_n = v_n$ for large n.

Since u_n and v_n are bounded in $H_0^1(\Omega)$, up to subsequences, there exist $u_*, v_* \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_*$ and $v_n \rightharpoonup v_*$ in $H_0^1(\Omega)$. By Rellich's embedding theorem, $u_n \to u_*$ and $v_n \to v_*$ in $L^2(\Omega)$. By assumption, we have $P_{E_2}u_* = P_{E_2}v_*$. By Proposition 1.1.7, u_* and v_* belong to $E_2 \setminus \{0\}$ and, here, $u_* = v_*$. As $u_* \in E_2 \setminus \{0\}$, let us remark that u_* is almost everywhere different from 0.

Observe that

$$\begin{cases} -\Delta(u_n - v_n) = a_n(x)(u_n - v_n), & \text{in } \Omega, \\ \qquad u_n - v_n = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.17}$$

where

$$a_n(x) := \begin{cases} \lambda_2 \dfrac{|u_n(x)|^{p_n-2}u_n(x) - |v_n(x)|^{p_n-2}v_n(x)}{u_n(x) - v_n(x)}, & \text{if } u_n(x) \neq v_n(x), \\ \lambda_2(p_n - 1)|u_n(x)|^{p_n-2}, & \text{if } u_n(x) = v_n(x). \end{cases}$$

Remarking that

$$a_n(x) = \lambda_2(p_n - 1)\int_0^1 |v_n(x) + s(u_n(x) - v_n(x))|^{p_n-2}\,\mathrm{d}s$$

and noting that

$$\left|\frac{|t|^{p-2}t - |s|^{p-2}s}{t - s}\right| \leqslant (p-1)(|t|^{p-2} + |s|^{p-2}),$$

we can apply the Lebesgue's dominated convergence theorem. It implies that, for every $q < +\infty$, $a_n \to \lambda_2$ in $L^q(\Omega)$. In particular, for every $\varepsilon > 0$ and n large enough, we have

$$\|a_n(x) - \lambda_2\|_{\frac{N}{2}} < \varepsilon.$$

Since $P_{E_2}(u_n - v_n) = 0$, Lemma 1.4.1 implies $u_n = v_n$ for large n. This concludes the proof. □

1.4.2 An abstract symmetry theorem

We now apply the previous uniqueness result to deduce partial symmetries of least energy nodal solution of Problem $(\mathscr{P}_p)$ when p is close to 2. The next Lemma is the key ingredient.

Lemma 1.4.3. *Let $M > 0$, $\overline{p}$ be given by Proposition 1.4.2 and $e_2 \in E_2 \setminus B(0, \frac{1}{M})$. If u_p is a solution of Problem $(\mathscr{P}_p)$ such that $P_{E_2} u_p = e_2$ and $u_p \in B_M$, $p < \overline{p}$, and if $T : H_0^1(\Omega) \to H_0^1(\Omega)$ is a continuous isomorphism satisfying the conditions:*

$$(i)\ T(E_2) = E_2; \quad (ii)\ T(E_2^\perp) = E_2^\perp; \quad (iii)\ Te_2 = e_2;$$
$$(iv)\ \text{for every } u \in H_0^1(\Omega),\ \mathscr{E}_p(Tu) = \mathscr{E}_p(u);$$

then $Tu_p = u_p$.

Proof. By Proposition 1.4.2, it is sufficient to prove that Tu_p is a solution of Problem $(\mathscr{P}_p)$ and $P_{E_2} Tu_p = P_{E_2} u_p = e_2$. On one hand, it follows from (iv) that, for all $v \in H_0^1(\Omega)$, $\langle \mathrm{d}\mathscr{E}_p(Tu_p), v\rangle = \langle \mathrm{d}\mathscr{E}_p(u_p), T^{-1}v\rangle$, so that we infer that Tu_p is a solution of Problem $(\mathscr{P}_p)$. On the other hand, since

$$P_{E_2} Tu_p + P_{E_2^\perp} Tu_p = Tu_p = TP_{E_2} u_p + TP_{E_2^\perp} u_p = P_{E_2} Tu_p + P_{E_2^\perp} Tu_p,$$

and the conditions (i) and (ii) ensure that $TP_{E_2} u_p \in E_2$ and $TP_{E_2^\perp} u_p \in E_2^\perp$, we deduce that $P_{E_2} Tu_p = TP_{E_2} u_p$. We then conclude the proof by using the condition (iii). □

Let G be a group whose identity is written 1 and inverse $(.)^{-1}$. Recall that a group action on $H_0^1(\Omega)$ is a continuous application $G \times H_0^1(\Omega) \to H_0^1(\Omega) : (g, u) \mapsto gu$ such that, for all $u \in H_0^1(\Omega)$ and for all $g, h \in G$,

$$\text{(i) } 1u = u, \quad \text{(ii) } (gh)u = g(hu), \quad \text{(iii) } u \mapsto gu \text{ is linear.}$$

Example 1.4.4. To any subgroup G of $O(N)$ such that $g(\Omega) = \Omega$ for every $g \in G$, one can associate the action

$$gu(x) := u(g^{-1}x).$$

Another action is given by

$$gu(x) := (\det g)u(g^{-1}x).$$

When $G = \{1, R\}$, where R is the reflection with respect to a hyperplane H, the fixed points of the action of G are, in the first case, the symmetric functions with respect to H, and, in the second case, the antisymmetric functions with respect to H.

The next theorem is a straightforward consequence of Lemma 1.4.3.

Theorem 1.4.5. *Let $(G_\alpha)_{\alpha\in E_2}$ be groups acting on $H_0^1(\Omega)$ in such a way that, for every $g \in G_\alpha$ and for every $u \in H_0^1(\Omega)$,*

$$\text{(i) } g(E_2) = E_2, \quad \text{(ii) } g(E_2^\perp) = E_2^\perp, \quad \text{(iii) } g\alpha = \alpha, \quad \text{(iv) } \mathscr{E}_p(gu) = \mathscr{E}_p(u).$$

Then, for all $M > 0$, if p is close enough to 2, any least energy nodal solutions $u_p \in \{u \in B(0,M) : P_{E_2}(u) \notin B(0, \frac{1}{M})\}$ of Problem ($\mathscr{P}_p$) belong to the invariant set of G_{α_p} where $\alpha_p := P_{E_2}u_p$.

It is worth pointing out that in particular, if G_α describes the symmetries (or antisymmetries) of α, we deduce that, for p close to 2, u_p respects the symmetries of its orthogonal projection α_p.

1.4.3 Example: the square

Theorem 1.4.6. *If Ω is a square, then, for p close to 2, least energy nodal solutions of Problem ($\mathscr{P}_p$) are odd with respect to the center of the square.*

Proof. Without loss of generality, we can work on square $(-1,1)^2$. The functions

$$v_1(x,y) = \cos(\tfrac{\pi}{2}x)\sin(\pi y), \qquad v_2(x,y) = \sin(\pi x)\cos(\tfrac{\pi}{2}y),$$

form an orthonormal basis of E_2. One directly checks that all the second eigenfunctions of $-\Delta$ are odd functions. They all belong to the invariant set of the group $G := \{1, -1\}$ acting in such a way that $(-1)u(x) = -u(-x)$. One concludes by Theorem 1.4.5. □

A nodal solution u of the problem $-\Delta u = u^3$ with Dirichlet boundary conditions on the square $(-1,1)^2$ is depicted in Figure 1.6 with different level lines. Figure 1.6 suggests that u is odd.

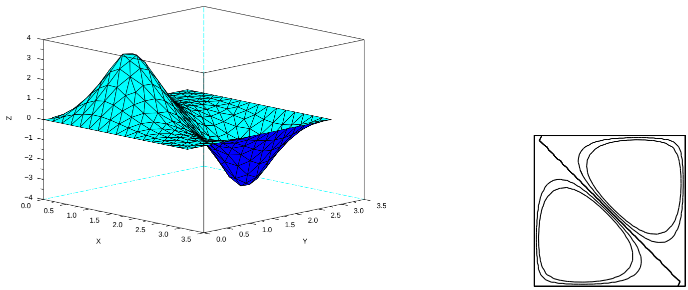


Figure 1.6: MMPA solution of the Lane–Emden Problem on a square.

Initial function	$\min u$	$\max u$	$\mathscr{E}_4(u^+)$	$\mathscr{E}_4(u^-)$
$(x+y-\frac{2}{5})(x+y-\frac{8}{5})(x-y\pm\frac{3}{5})$	-5.2	5.2	23.9	23.9

Table 1.3: Characteristics of a l.e.n.s. on the square.

Moreover, the nodal line of u seems to be a diagonal, u seems to be antisymmetric with respect to that diagonal and symmetric with respect to the other diagonal. Further numerical computations confirm our guess. Indeed, for $\theta \in \mathbb{R}$, set

$$v_\theta = \cos\theta\, v_1 + \sin\theta\, v_2$$

and consider the function

$$H_*(\theta) := \sup_{t>0} \mathscr{E}_*(t v_\theta),$$

where $\mathscr{E}_*$ is defined in Proposition 1.1.3. Since $\int_\Omega |v_\theta|^2 = 1$, one easily computes

$$S(\theta) := \log \frac{2}{\lambda_2} H_*(\theta) = -\int_\Omega |v_\theta|^2 \log|v_\theta|^2.$$

This function can be thought as the entropy associated to the density $|v|^2$, and the accumulation points of the minimal energy nodal solutions are the eigenfunctions with minimal entropy. The numerical computation of S, see Fig-

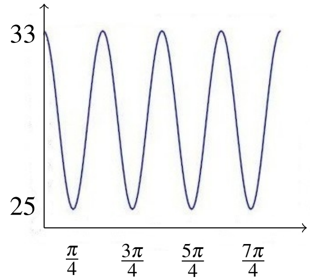


Figure 1.7: Function $S : [0,2\pi] \to \mathbb{R} : \theta \mapsto S(\theta)$.

ure 1.7, suggests that these eigenfunctions are the functions v_θ with $\theta = \frac{\pi}{4} + k\frac{\pi}{2}$. Up to rotations of $k\frac{\pi}{2}$, these can be written as

$$4\cos\tfrac{\pi}{2}x\,\cos\tfrac{\pi}{2}y\,\cos\tfrac{\pi}{2}(x-y)\,\sin\tfrac{\pi}{2}(x+y).$$

The computations of the entropy 1.7 has been performed using the composite Simpson method. The step is $\mu = 1/1024$. The software is Matlab. About the approximation error, on one hand, we have the estimate

$$|\nabla(\cos(\theta)v_1 + \sin(\theta)v_2)| \leqslant \pi\sqrt{5} \leq 12$$

and the function takes its values in $[-2,2]$, whereas, on the other hand, it is straightforward that

$$|t^2\log t^2 - s^2\log s^2| \leqslant (4(1+\log 4)(t-s) \leqslant 12(t-s),$$

where $4(1+\log 4)$ is a Lipschitz constant for the function $t^2\log t^2$ on $[-2,2]$. So, we therefore infer that the approximation error is bounded by

$$\frac{4(1+\log 4)\pi\sqrt{5}}{2048} = 0.0327. \tag{1.18}$$

To finish this section, in Figure 1.8, we use MMPA on the equation $-\Delta u = \lambda_2|u|u$ with DBC to approach solutions in $H_0^1(\Omega)$ restricted to odd functions with respect to a diagonal (resp. to a median). It is expected that solutions are with least energy. We can remark in Tables 1.4 and 1.5 that the energy of the "diagonal" solution is strictly less than the "median" one. A numerical proof is in progress in collaboration with P. J. McKenna and C. Troestler.

These computations and Theorem 1.1.8 legitimate the following conjecture.

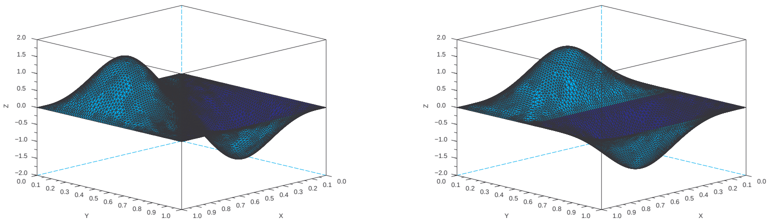


Figure 1.8: Difference between diagonal and median solutions.

symmetry	initial function
Diagonal	$\sin(\pi x)\sin(2\pi y)+\sin(2\pi x)\sin(\pi y)$
Median	$\sin(\pi x)\sin(2\pi y)$

Table 1.4: Equation $-\Delta u = \lambda_2 |u| u$.

	$\min u$	$\max u$	$\mathscr{E}_3(u)$	$\|u\|$
Diagonal	-1.506	1.506	3.1052	4.3163
Median	-1.5354	1.5354	3.5162	4.5931

Table 1.5: Characteristics of the two approximate solutions on the square.

Conjecture 1.4.7. If Ω is a square, then, for p close to 2, any least energy nodal solution of Problem $(\mathscr{P}_p)$ is symmetric with respect to one diagonal and antisymmetric with respect to the orthogonal one.

1.5 Symmetry breaking

In the previous Section 1.4.3, the study of accumulation points of least energy nodal solutions $(u_p)_{p>2}$ allowed to exhibit (at least numerically) the symmetries of least energy nodal solutions for p close to 2 on the square. In the same way, under the assumption that the minima of the entropy function S are not given by "median" functions (as suggested on Figure 1.7), this analysis can also be used to obtain some symmetry breaking for larger p. We begin by studying small perturbations of the Laplacian.

1.5.1 Perturbation of the Laplacian operator

The previous results, related to general domains, can be extended to Problem ($\tilde{\mathscr{P}}_p$)

$$\begin{cases} -\operatorname{div}(A_p \nabla u) = \lambda_2 |u|^{p-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$\tilde{\mathscr{P}}_p$}$$

with $A_p \in \mathscr{C}(\Omega, M^{N\times N})$, where $M^{N\times N}$ is the set of symmetric $N \times N$ matrices, is such that $A_2 = \mathrm{id}$ and $p \mapsto A_p$ is uniformly differentiable at $p = 2$ (i.e. there exists a matrix $A_2' \in M^{N\times N}$ verifying

$$\left\| \frac{A_p - A_2 - A_2'(p-2)}{p-2} \right\|_\infty \to 0, \text{ as } p \to 2).$$

We are interested in critical points of the functional

$$\tilde{\mathscr{E}}_p : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2} \int_\Omega (A_p \nabla u) \cdot \nabla u - \frac{\lambda_2}{p} \int_\Omega |u|^p.$$

Theorem 1.5.1. *Let $(u_p)_{p>2}$ be least energy nodal solutions of Problem ($\tilde{\mathscr{P}}_p$). If $u_{p_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$ for some sequence $p_n \to 2$, then $u_{p_n} \to u_* \neq 0$ in $H_0^1(\Omega)$, u_* satisfies*

$$\begin{cases} -\Delta u_* = \lambda_2 u_*, & \text{in } \Omega, \\ u_* = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$\tilde{\mathscr{E}}_*(u_*) = \inf\{\tilde{\mathscr{E}}_*(u) : u \in E_2 \setminus \{0\}, \langle \mathrm{d}\mathscr{E}_*(u), u\rangle = 0\},$$

where

$$\tilde{\mathscr{E}}_* : E_2 \to \mathbb{R} : u \mapsto \int_\Omega (A_2' \nabla u) \cdot \nabla u + \frac{\lambda_2}{2}(u^2 - u^2 \log u^2).$$

Sketch of the proof. All the arguments developed in Section 1.1 hold for $(\tilde{\mathscr{P}}_p)$, up to some small differences. Indeed, the equations (1.6) and (1.7) become respectively

$$\begin{cases} -\Delta w - \lambda_2 w = \operatorname{div}(A_2' \nabla u_*) + \lambda_2 u_* \log |u_*|, \\ P_{E_2} w = 0, \end{cases}$$

and

$$\begin{aligned} \int_\Omega |\nabla v_p^\pm|^2 &= \lambda_2 \int_\Omega (v_p + (p-2) u_* \log |u_*|) v_p^\pm - (p-2) \int_\Omega (A_2' \nabla u_*) \cdot \nabla v_p^\pm \\ &= \lambda_2 \int_\Omega |v_p^\pm|^p - (p-2) \int_\Omega (A_2' \nabla u_*) \cdot \nabla v_p^\pm + o(p-2). \end{aligned}$$

The computation of $t_p^\pm$ gives

$$(t_p^\pm)^{p-2} = \frac{(A_p \nabla v_p^+) \cdot \nabla v_p^+}{\lambda_2 \|v_p^\pm\|_p^p} = \frac{\|\nabla v_p^\pm\|_2^2 + (p-2) \int_\Omega \left(\frac{A_p - \mathrm{id}}{p-2} \nabla v_p \right) \cdot \nabla v_p^\pm}{\lambda_2 \|v_p^\pm\|_p^p},$$

and the proof that $t_p^\pm \to 1$ follows from the fact that

$$\begin{aligned} &(p-2) \textstyle\int_\Omega \left(\frac{A_p - \mathrm{id}}{p-2} \nabla v_p \right) \cdot \nabla v_p^\pm - (p-2) \int_\Omega (A_2' \nabla u_*) \cdot \nabla v_p^\pm \\ &= (p-2) \textstyle\int_\Omega \left(\left(\frac{A_p - \mathrm{id}}{p-2} - A_2' \right) \nabla v_p \right) \cdot \nabla v_p^\pm + (p-2)^2 \int_\Omega (A_2' \nabla w) \cdot \nabla v_p^\pm \\ &= o(p-2), \end{aligned}$$

giving then the same conclusion as for Problem $(\mathscr{P}_p)$.

In a similar fashion, all the other results extend to the present framework. □

Observe that it is enough to have the differentiability along the sequences.

Going to the abstract symmetry results, Proposition 1.4.2 about the uniqueness of the solution with a given projection on E_2 extends immediately to Problem $(\tilde{\mathscr{P}}_p)$.

Proposition 1.5.2. *For every $M > 0$, there exists $\bar{p} > 2$ such that, for every $p \in (2, \bar{p})$, if $u_p, v_p \in \{u \in B(0,M) : P_{E_2} \notin B(0, \frac{1}{M})\}$ solve the boundary value problem*

$$\begin{cases} -\operatorname{div}(A_p \nabla u) = \lambda_2 |u|^{p-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

then $P_{E_2} u_p = P_{E_2} v_p$ implies $u_p = v_p$.

Theorem 1.5.3. *Let $(G_\alpha)_{\alpha\in E_2}$ be groups acting on $H_0^1(\Omega)$ in such a way that, for every $g \in G_\alpha$ and for every $u \in H_0^1(\Omega)$,*

$$g(E_2) = E_2, \quad g(E_2^\perp) = E_2^\perp, \quad g\alpha = \alpha \quad \text{and} \quad \tilde{\mathscr{E}}_p(gu) = \tilde{\mathscr{E}}_p(u).$$

Then, for p close to 2, any least energy nodal solutions u_p of Problem ($\tilde{\mathscr{P}}_p$) belongs to the invariant set of G_{α_p} where $\alpha_p := P_{E_2} u_p$.

1.5.2 Symmetry breaking on rectangles

In this section, we prove that, for some rectangles close to a square, least energy nodal solutions cannot be odd nor even with respect to a median (see Figure 1.9). On a rectangle R_ε with sides of lengths 2 and $(1+\varepsilon)2$, we consider the problem

$$\begin{cases} -\Delta u = \lambda_2 |u|^{p-2}u, & \text{in } R_\varepsilon, \\ u = 0, & \text{on } \partial R_\varepsilon. \end{cases} \tag{$\mathscr{P}_\varepsilon$}$$

The change of variable $\tilde{u}(x,y) = u(x,(1+\varepsilon)y)$, leads to the equivalent problem on the square $Q = (-1,1)^2$

$$\begin{cases} -\partial_x^2 \tilde{u} - (1+\varepsilon)^{-2}\partial_y^2 \tilde{u} = \lambda_2 |\tilde{u}|^{p-2}\tilde{u}, & \text{in } Q, \\ \tilde{u} = 0, & \text{on } \partial Q. \end{cases}$$

In both problems, $\lambda_2 = \lambda_2(Q)$ for $-\Delta$. Observe also that u is a least energy nodal solution of Problem ($\mathscr{P}_\varepsilon$) if and only if $\tilde{u}$ is a least energy nodal solution on the square Q.

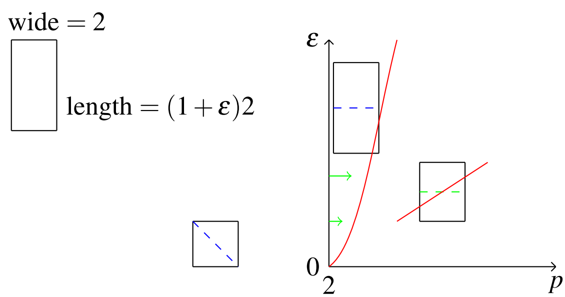


Figure 1.9: Symmetry breaking: rectangles → square.

We discuss the existence of $C > 0$ and $2 < \bar{p}$ such that if $2 < p < \bar{p}$ and $|\varepsilon| \leqslant C(p-2)$, then every least energy nodal solution of Problem ($\mathscr{P}_\varepsilon$) is neither symmetric nor antisymmetric with respect to one of the medians of R_ε. If not, by contradicion, there exists $p_n \to 2$ and $\varepsilon_n = o(p_n - 2)$ such that Problem ($\mathscr{P}_\varepsilon$) has a solution u_n that is symmetric or antisymmetric to one of the medians. Define $\tilde{u}_n$ by the change of variables given above. The functions $\tilde{u}_n$ are least energy nodal solutions of Problem ($\tilde{\mathscr{P}}_p$) with $\Omega = Q$ and

$$A_{p_n} = \mathrm{id} - \big(1 - (1+\varepsilon_n)^{-2}\big) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $A_2' = 0$, the corresponding functional $\tilde{\mathscr{E}}_*$ reads as $\mathscr{E}_*$ for Problem ($\mathscr{P}_p$). In particular, the accumulation points of the sequence are symmetric or antisymmetric with respect to one of the medians. Such eigenfunctions cannot be minimizers of $\tilde{\mathscr{E}}_*$ on $\tilde{\mathscr{N}}_*$ as numerical computations illustrated by Figure 1.7 show (one would indeed conjecture that they are maximizers on $\tilde{\mathscr{N}}_*$). So, up to a computer assisted proof showing exactly that the minimizers are not "median functions", we obtain the following result.

Theorem 1.5.4. *There exists $C > 0$ and $2 < \bar{p}$ such that if $2 < p < \bar{p}$ and $|\varepsilon| \leqslant C(p-2)$, then every least energy nodal solution of Problem ($\mathscr{P}_\varepsilon$) is neither symmetric nor antisymmetric with respect to one of the medians of R_ε.*

Now, consider the case where $\varepsilon = \gamma(p-2)$. Arguing as in Section 1.4.3, we can compute

$$\tilde{H}(\theta) = \sup_{t>0} \tilde{\mathscr{E}}_*(t\theta)$$

and, as $A_2' = -2\gamma \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, one obtains

$$\begin{aligned} \tilde{S}_\gamma(\theta) &:= \log \frac{2}{\lambda_2} \tilde{H}(\theta) \\ &= S(\theta) - \frac{4\gamma}{\lambda_2} \int_Q (\cos\theta\, \partial_y v_1 + \sin\theta \partial_y v_2)^2 \\ &= S(\theta) - \frac{4\gamma}{5}(3\cos^2\theta + 1), \end{aligned}$$

where S is defined page 57.

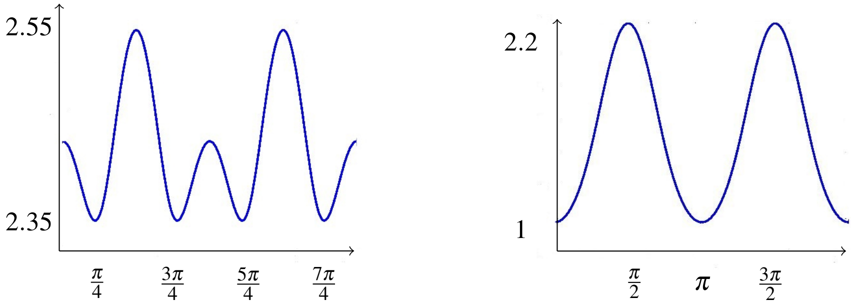


Figure 1.10: $\tilde{S}_\gamma$ for $\gamma = 0.05$ and 0.5.

The function S achieves a global minimum at $\frac{\pi}{4} + k\frac{\pi}{2}$, see Figure 1.7, while the function $\frac{-4\gamma}{5}(3\cos^2\theta + 1)$ achieves its global minimum at $k\pi$. Since these points are critical points of S, the global minimizers of $\tilde{S}_\gamma(\theta)$ are $k\pi$ when γ is large enough. The functions v_θ for these values of θ being symmetric with respect to the medians of Q, we obtain the following conjecture.

Conjecture 1.5.5. There exists $c > 0$ and $\bar{p} > 2$, such that if $\varepsilon > c(p-2)$ and $2 < p < \bar{p}$, then every least energy nodal solutions of Problem $(\mathscr{P}_\varepsilon)$ is symmetric with respect to the longest median and antisymmetric with respect to the shortest one.

The shapes of the graphs of $\tilde{S}_{0.05}$ and $\tilde{S}_{0.5}$ (see Figure 1.10) seem to indicate that the treshold value for γ corresponds to a degenerate minima. Hence, it is expected that the first γ for which $k\pi$ is a global minimum should verify $\tilde{S}''_\gamma(k\pi) = S''(k\pi) + \frac{24}{5}\gamma = 0$. A numerical approximation of the second derivative of S gives

$$\gamma = 0.2167.$$

We conclude this section by numerically illustrating the symmetry breaking. Consider a rectangle of sidelengths 1 and 1.2. Figure 1.11 shows a nodal solution u of the problem $-\Delta u = u^3$ with Dirichlet boundary conditions obtained by MMPA.

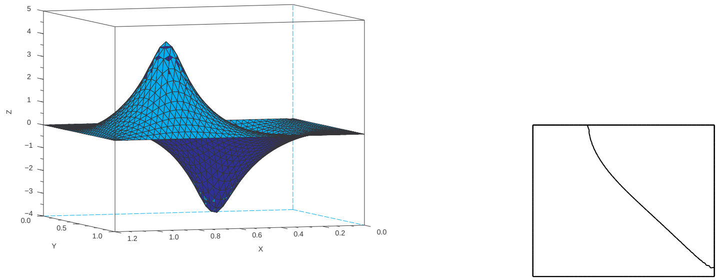


Figure 1.11: Non-symmetric solution for Lane–Emden Problem on a rectangle.

Initial function	$\min u$	$\max u$	$\mathscr{E}_4(u^+)$	$\mathscr{E}_4(u^-)$
$\cos(\pi x)\cos(\pi y)\, x(x-1.2)\, y(y-1)$	-3.7	4.2	19.7	26.2

Table 1.6: Characteristics of a non-symmetric solution on a rectangle.

1.6 What about ground state solutions?

To finish this chapter, we study symmetries of ground state solutions. By using "moving planes" method defined by Gidas, Ni and Nirenberg [32], we directly obtain that, for any $2 < p < 2^*$, a ground state solution respects symmetries of the domain Ω. Nevertheless, this technique does not work for any domain. In particular, we need Ω convex (and "smooth") to use it.

With our "general domains" technique defined in Section 1.4, we obtain, for p close to 2 and on possibly non-convex domains, an alternative to the "moving planes" method.

Clearly, our technique can very easily be modified for ground state solutions. About the asymptotic behavior, this time, the good rescaling is given by λ_1, the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ with DBC. Now, we are working with the problem

$$\begin{cases} -\Delta u = \lambda_1 |u|^{p-2}u, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{$G\mathscr{P}_p$}$$

A function u is a solution of Problem ($\mathscr{P}_p$) if and only if $\lambda_1^{1/(2-p)} u$ is a solution of ($G\mathscr{P}_p$). We obtain, using the same kind of arguments as in Section 1.1 (in fact, as we just need to work in $\mathscr{N}_p$, the arguments are easier), that a family of ground state solutions is bounded in $H_0^1(\Omega)$ and stays away from 0. Moreover, we obtain the following result for Problem (LEP).

Theorem 1.6.1. *If $(u_p)_{p>2}$ is a family of ground state solutions for Problem (LEP), then there exists a real $C > 0$ such that*

$$\|u_p\| \leqslant C\lambda_1^{\frac{1}{p-2}}.$$

If $\lambda_1^{\frac{1}{2-p_n}} u_{p_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$ for some sequence $p_n \to 2$, then $\lambda_1^{\frac{1}{2-p_n}} u_{p_n} \to u_* \neq 0$ in $H_0^1(\Omega)$, u_* satisfies

$$\begin{cases} -\Delta u_* = \lambda_1 u_*, & \text{in } \Omega, \\ u_* = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$\mathscr{E}_*(u_*) = \inf\{\mathscr{E}_*(u) : u \in E_1 \setminus \{0\}, \langle \mathrm{d}\mathscr{E}_*(u), u\rangle = 0\},$$

where

$$\mathscr{E}_* : E_1 \to \mathbb{R} : u \mapsto \frac{\lambda_1}{2}\int_\Omega u^2 - u^2 \log u^2.$$

Concerning symmetries, on general domains, by working with λ_1 and E_1, we can work with the technique introduced in Section 1.4. Like this, we obtain the following result.

Theorem 1.6.2. *For p close to 2, ground state solutions of Problem (LEP) respect symmetries of its projection on E_1.*

In particular, on annuli, we obtain that, for p close to 2, ground state solutions must be radial, which is not the case for large p (see e.g. [20]). Nevertheless, it seems that the symmetry breaking occurs very fast. In fact, by using MPA, we already have that the ground state solutions of problem $-\Delta u = |u|u$ with DBC do not seem radial (see Figure 1.12). Let us define $r = \sqrt{x^2 + y^2}$ on Table 1.7.

Certainly, as $\dim E_1 = 1$, we also can use the "implicit function" method defined in Section 1.2.1. We obtain the uniqueness up to a multiplicative factor by -1, for p close to 2, of ground state solutions.

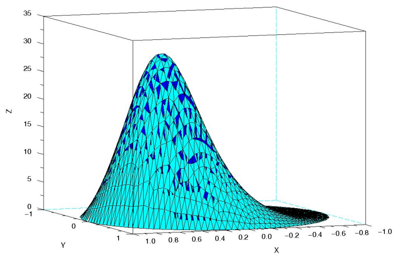


Figure 1.12: MPA solution on an annulus.

Initial function	$\min u$	$\max u$	$\mathscr{E}_3(u^+)$	$\mathscr{E}_3(u^-)$
$\cos((0.5\pi)r)\cos((5\pi)r)$	0.0	30.76	741.22	0.0

Table 1.7: Characteristics of the ground state on an annulus.

Chapter 2

Nodal line structure of least energy nodal solutions

In this chapter, we are interested in the nodal line structure of least energy nodal solutions for the Lane–Emden Problem with DBC ($\mathscr{P}_p$)

$$\begin{cases} -\Delta u = \lambda_2 |u|^{p-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^2$ is an open bounded connected domain and $p > 2$. In Chapter 1, we studied some symmetries of solutions on rectangles, balls, annuli, squares,... In particular, for these domains, we obtained that the nodal line of least energy nodal solutions intersects $\partial\Omega$, at least for p close to 2. We would like to know, for which domains Ω not necessarily symmetric and $2 < p < 2^*$, the nodal line of a least energy nodal solution intersects $\partial\Omega$. In Theorem 1.1.8, it is proved that a family of least energy nodal solutions $(u_p)_{p>2}$ of Problem ($\mathscr{P}_p$) is bounded in $H_0^1(\Omega)$ and stays away from the zero function. Moreover, weak accumulation points for the H_0^1-norm are in fact strong accumulation points belonging to E_2. A H_0^1-convergence is clearly not enough to control the level curves and to solve the problem. Here, we establish that $(u_p)_{p>2}$ converges, in a suitable sense, to some $u_* \in E_2$ in a such a manner that the zero set of u_p is close to the zero set of u_*. We obtain that the nodal line intersects the boundary when

Ω is convex. For Ω non-convex, the nodal line may well be strictly included in $\partial\Omega$.

In the same vein, let us just remark that the control of the zero set can be an interesting tool to obtain some symmetry breaking on symmetric domains where eigenfunctions of $-\Delta$ are not symmetric.

This job is inspired by the paper [38] written with C. Troestler.

2.1 Convergence in $\mathscr{C}^1(\Omega)$

To start, let us just recall that the classical norm of a function u in $\mathscr{C}^n(\Omega)$, for $n \in \mathbb{N}$, is given by

$$\|u\|_{\mathscr{C}^n} := \sup_{x\in\Omega} \sum_{|\alpha|\leq n} |\partial^\alpha u(x)|,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \sum_{i=1}^n \alpha_i$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ and, for any j, $\partial_{x_j}^0 = \mathrm{id}$. During this section, we use different elliptical regularity results. The reader can find them in [19, 22, 33].

Lemma 2.1.1. *Let Ω be a bounded open domain of class $\mathscr{C}^k$, for $k \geqslant 2$. There exists a constant $C \in \mathbb{R}$ such that if $u \in W^{1,2}(\Omega)$ verifies $-\Delta u = f^*$, where $f^* \in W^{k-2,q}(\overline{\Omega})$[1], then $u \in W^{k,q}(\overline{\Omega})$ and $\|u\|_{W^{k,q}(\overline{\Omega})} \leqslant C\|f^*\|_{W^{k-2,q}(\overline{\Omega})}$.*

Lemma 2.1.2. *The Sobolev space $W^{k,q}(\Omega) \Subset \mathscr{C}^{m,\alpha}(\overline{\Omega})$, for $m \in \mathbb{N}$ strictly less than $k - \frac{N}{q}$, where N is the dimension and $0 \leqslant \alpha < \lfloor k - \frac{N}{q} - m \rfloor$.*

Lemma 2.1.3. *For any sequence $(p_n) \subseteq (2, +\infty)$ and $u_* \in \mathscr{C}(\overline{\Omega})$, if $p_n \to 2$ and $u_{p_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$, then $|u_{p_n}|^{p_n-2}u_{p_n} \to u_*$ in $L^q(\Omega)$, for all $1 < q < 2^*$.*

Proof. Let $1 < q < 2^*$. There exists $n_0 \in \mathbb{N}$ s.t., by denoting $\bar{p} := \sup_{n\geq n_0} p_n$, we have $1 < r := q(\bar{p} - 1) < 2^*$. Given the classical Sobolev embedding theorem, one can assume $u_{p_n} \to u_*$ in $L^r(\Omega)$. Thus, taking if necessary a subsequence, there exists $g \in L^r(\Omega)$ such that, almost everywhere, $|u_{p_n} - u_*| \leqslant g$ (see result A.2). As $u_* \in \mathscr{C}(\bar{\Omega})$, there exists a constant C such that $|u_{p_n}| \leqslant g + C$ almost everywhere. Therefore, for all $n \in \mathbb{N}$, as the eigenfunctions of $-\Delta$ belong to $\mathscr{C}^2(\Omega) \cap \mathscr{C}(\overline{\Omega})$, we can assume that

$$\big||u_{p_n}|^{p_n-2}u_{p_n} - u_*\big| \leqslant \max(1, g + C)^{\bar{p}-1} + C \in L^{r/(\bar{p}-1)}(\Omega).$$

[1] $W^{k-2,q}(\Omega) := \{u \in L^q(\Omega) : (k-2)^{th}$ weak derivative of u belongs to $L^q(\Omega)\}$.

Thus, using Lebesgue's dominated convergence theorem and the fact that the limit does not depend on the subsequence, we can conclude. $\square$

Proposition 2.1.4. *Let $\Omega \subseteq \mathbb{R}^2$ be a domain of class $\mathscr{C}^2$ and $(u_p)_{p>2}$ be a family of solutions for Problem $(\mathscr{P}_p)$. If $u_* \in \mathscr{C}(\overline{\Omega})$ is a weak accumulation point of $(u_p)_{p>2}$ in $H_0^1(\Omega)$ then u_* is an accumulation point in $\mathscr{C}^1(\overline{\Omega})$.*

Proof. By hypothesis, there exists a sequence $p_n \to 2$ such that $u_{p_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$. We have, for all $n \in \mathbb{N}$,

$$\begin{cases} -\Delta(u_{p_n} - u_*) = \lambda_2 \left(|u_{p_n}|^{p_n-2} u_{p_n} - u_* \right), & \text{in } \Omega, \\ \qquad u_{p_n} - u_* = 0, & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.1.3, $|u_{p_n}|^{p_n-2} u_{p_n} \to u_*$ in $L^q(\Omega)$, for all $1 < q$. Result 2.1.1 implies that, for p_n sufficiently close to 2, $u_{p_n} \in W^{2,q}(\Omega)$ and $u_{p_n} \to u_*$ in $W^{2,q}(\Omega)$. By Lemma 2.1.2, we know that $W^{2,q}(\Omega) \Subset \mathscr{C}^{m,\alpha}(\bar{\Omega})$ when $m < 2 - 2/q$ and $0 \leqslant \alpha < \lfloor 2 - \frac{2}{q} - m \rfloor$. Thus, taking q sufficiently large so that $W^{2,q} \Subset \mathscr{C}^1(\overline{\Omega})$, we conclude that $u_{p_n} \to u_*$ in $\mathscr{C}^1(\overline{\Omega})$. $\square$

Remark 2.1.5.

- By Chapter 1, we know that the accumulation points of a family of least energy nodal solutions belong to $E_2 \setminus \{0\}$. So, the previous Proposition 2.1.4 can be used in that case.

- Proposition 2.1.4 is true if we consider a family of ground state solutions. One just needs to replace λ_2 (resp. E_2) by λ_1 (resp. E_1).

- The above results hold directly in dimensions $N < 2^*$ (i.e. $N = 2$ or 3).

- To obtain only $\mathscr{C}$-convergence, previous results require a smooth domain and $N < 2 \cdot 2^*$ (i.e. $2 \leqslant N \leqslant 5$).

- Lemmas 2.1.1 and 2.1.2 can be extended to a product of intervals (see e.g. [19]). Moreover, our $\mathscr{C}$-convergence can be extended to any dimensions. Indeed, by a classical bootstrap, if $u_p \to u_*$ in $H_0^1(\Omega)$ and $(u_p)_{p>2}$ is a family of solutions for Problem $(\mathscr{P}_p)$, $u_p \to u_*$ in $L^q(\Omega)$ for any $1 < q < +\infty$. So, we can improve Lemma 2.1.3 to obtain that $|u_p|^{p-2}u_p \to u_*$ for any $1 < q < +\infty$. By working in Proposition 2.1.4, we obtain the $\mathscr{C}$-convergence.

- Concerning the previous bootstrap, if u_p is a solution of Problem ($\mathscr{P}_p$), we have that $u_p \in L^{+\infty}(\Omega)$ and $\|u_p\|_\infty \leq C\|u_p\|^{n(p-1)}$ with C and n independent of p.

2.2 Nodal line structure

2.2.1 Asymptotic results

In this section, we call the ε-neighbourhood of a set A the set of points $x \in \Omega$ such that the distance between x and A is less than ε. For any $u \in \mathscr{C}(\Omega)$ with two nodal domains, let $\mathscr{N}(u) := \overline{\{x \in \Omega \mid u = 0\}}$ denotes its nodal set, ND_u^+ its positive nodal domain, and ND_u^- its negative nodal domain. First, we mention a classical well-known result in the literature as the Höpf's Lemma. The reader can for example find a proof in [35, 48, 59].

Lemma 2.2.1. *If $\Delta u \leq 0$ on a connected open bounded domain Ω, the minimum of u is obtained at $x \in \partial\Omega$ and Ω satisfies the "inner ball" condition[2] at x then the external normal derivative $\partial_\nu u$ is negative at x.*

The following propositions 2.2.2 and 2.2.3 prove that, for p close to 2, we can control nodal sets of least energy nodal solutions if we know these of u_*.

Proposition 2.2.2. *Let Ω be a domain of class $\mathscr{C}^2$ and $u_* \in E_2$ be such that $\overline{\mathrm{ND}_{u_*}^+} \setminus \mathscr{N}(u_*)$ and $\overline{\mathrm{ND}_{u_*}^-} \setminus \mathscr{N}(u_*)$ intersect the same connected component of $\partial\Omega$. If $u_{p_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$, then, for n large, $\mathscr{N}(u_{p_n})$ intersects $\partial\Omega$.*

Proof. By contradiction, let us assume that there exists a subsequence, still denoted p_n, such that $p_n \to 2$ and the nodal sets of u_{p_n} do not intersect $\partial\Omega$. Let Γ be a connected component of $\partial\Omega$ that both $\overline{\mathrm{ND}_{u_*}^+} \setminus \mathscr{N}(u_*)$ and $\overline{\mathrm{ND}_{u_*}^-} \setminus \mathscr{N}(u_*)$ intersect. Since $\mathscr{N}(u_{p_n})$ stays away from Γ, u_{p_n} has always the same sign in a neighbourhood of Γ. Going if necessary to a subsequence, we can assume that $u_{p_n} > 0$ in a neighbourhood of Γ for all n (the case $u_{p_n} < 0$ can be treated similarly). Höpf's Lemma 2.2.1 implies $\partial_\nu u_{p_n} < 0$ for all $x \in \Gamma$. Pick $x \in \Gamma \cap \left(\overline{\mathrm{ND}_{u_*}^-} \setminus \mathscr{N}(u_*)\right)$. Since $\mathscr{N}(u_*)$ is compact, there exists a connected neighbourhood U of x such that $u_* < 0$ in $U \cap \Omega$. Thus, by Hopf's Lemma 2.2.1,

[2] There exists a ball $B \subseteq \Omega$ such that $x \in \partial B$.

$\partial_\nu u_* > 0$. As $u_{p_n} \to u_*$ in $\mathscr{C}^1(\overline{\Omega})$ by Proposition 2.1.4, this is a contradiction. $\square$

Proposition 2.2.3. *Let Ω be an open bounded connected domain and assume it is of class $\mathscr{C}^2$. If $u_{p_n} \rightharpoonup u_*$ and the nodal line of u_* does not intersect $\partial\Omega$, then, for n large, the nodal line of u_{p_n} does neither intersect $\partial\Omega$.*

Proof. As $\mathscr{N}(u_*)$ does not intersect $\partial\Omega$ which is compact, there exists $\varepsilon > 0$ such that u_* does not vanish in an ε-neighbourhood of $\partial\Omega$. Thus Hopf's Lemma 2.2.1 and the compactness of $\partial\Omega$ imply the existence of a constant $C > 0$ such that $|\partial_\nu u_*(x)| \geqslant C$ for all $x \in \partial\Omega$. Using the fact that $u_{p_n} \to u_*$ in $\mathscr{C}^1(\overline{\Omega})$ (Proposition 2.1.4), we conclude that, for p_n sufficiently close to 2, $|\partial_\nu u_{p_n}| \geqslant C/2$ on $\partial\Omega$ and therefore u_{p_n} does not vanish in a neighbourhood of $\partial\Omega$. $\square$

Remark 2.2.4. The boundary of a simply connected domain of $\mathbb{R}^2$ has a unique connected component [43].

2.2.2 The convex case

In 1994, G. Alessandrini [5] showed that, when Ω is convex, the nodal line of the second eigenfunctions of $-\Delta$ always intersects $\partial\Omega$ at exactly 2 points. Using this and Proposition 2.2.2, we immediately obtain the following result.

Theorem 2.2.5. *On a convex domain of class $\mathscr{C}^2$, for p close to 2, the nodal line of least energy nodal solutions u_p intersects the boundary of Ω.*

2.2.3 The non-simply connected case

M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili [40] exhibited a connected (but non-simply connected) domain Ω such that the second eigenvalue of $-\Delta$ is simple and the nodal line of a second eigenfunction does not intersect $\partial\Omega$. To describe it, let us work in polar coordinates $x = r(\cos\omega, \sin\omega)$ with $0 \leqslant \omega < 2\pi$. Select $0 < R_1 < R_2 < R_3$ such that

$$\lambda_1\big(B(0,R_1)\big) < \lambda_1\big(A(R_2,R_3)\big) < \lambda_2\big(B(0,R_1)\big),$$

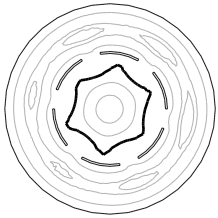


Figure 2.1: Nodal line (in bold) of u_*, for 6 bridges.

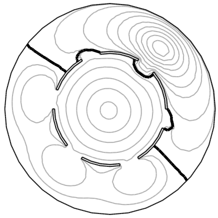


Figure 2.2: Nodal line (in bold) of u_3, for 6 bridges.

where $A(R_2,R_3)$ denotes the annulus $B(0,R_3)\setminus B[0,R_2]$. Then consider

$$\begin{aligned}D_{b,\varepsilon} := {} & B(0,R_1)\cup A(R_2,R_3)\\ & \cup \bigcup_{j=0}^{b-1}\left\{x\in\mathbb{R}^2 : R_1\leqslant r\leqslant R_2,\ \left|\omega-\frac{2\pi j}{b}\right| \bmod 2\pi<\varepsilon\right\},\end{aligned}$$

which is a disc surrounded by an annulus joined by b small bridges (see figures 2.1 and 2.2). For b sufficiently large and ε sufficiently small, M. Hoffmann-Ostenhof and al. show that the nodal line of u_* does not intersect $\partial D_{b,\varepsilon}$. Since their proof does not use the structure of the bridges between the ball $B(0,R_1)$ and the annulus $A(R_2,R_3)$ but only the group of reflections that leaves $D_{b,\varepsilon}$ invariant and the fact that the bridges are small, we can smooth the domain so that it is of class $\mathscr{C}^2$. By Proposition 2.2.3, we then conclude as follows.

Theorem 2.2.6. *There exists connected domains such that, for p close to 2, the nodal sets of the least energy nodal solutions do not intersect the boundary of Ω.*

Figure 2.1 shows the level curves of a second eigenfunction of $-\Delta$ for $b = 6$ bridges. We can see that the nodal line is included into the ball $B(0, R_1)$ and that the second eigenfunction u_* is even with respect to any reflection leaving Ω invariant [40]. Therefore, by results given in Chapter 1, u_p is even with respect to any reflection that leaves Ω invariant, for p close to 2.

Numerical experiments seem to indicate that Theorem 2.2.6 and the above symmetry properties of u_p do not remain valid for values of p farther from 2. For example, Figure 2.2 represents level curves of u_p for $p = 3$. It clearly shows that $\mathcal{N}(u_3)$ touches $\partial\Omega$ and that u_3 is only even with respect to a single reflection.

Chapter 3

Schrödinger and q-Laplacian Lane–Emden problem

From Chapter 3 *to Chapter* 5*, we generalize results presented in Chapter* 1*. We would like to know whether asymptotic and symmetry results are working for more general problems. Concerning symmetry (or non-symmetry) results, we will essentially focus on the "general method" studied in Section 1.4 in order to study general domains and obtain some symmetry breaking. In this chapter, we work with some perturbations of the linear part* Δ *of Problem* ($\mathscr{P}_p$)*. Let us just remark that, in Chapter* 1 *(see Section 1.5), we already proved that asymptotic and symmetry results are valid for Problem* ($\tilde{\mathscr{P}}_p$)

$$\begin{cases} -\operatorname{div}(A_p \nabla u) = \lambda |u|^{p-2} u, & \textit{in } \Omega, \\ u = 0, & \textit{on } \partial\Omega, \end{cases}$$

with $A_p \in \mathscr{C}(\Omega, M^{N\times N})$*, where* $M^{N\times N}$ *is the set of symmetric* $N \times N$ *matrices, such that* $A_2 = \mathrm{id}$ *and* $p \mapsto A_p$ *is uniformly differentiable at* $p = 2$*.*

In Section 3.1, we are studying the superlinear elliptic boundary value Problem ($S\mathscr{P}_p$) *of Schrödinger type*

$$\begin{cases} -\Delta u + V u = \lambda |u|^{p-2} u, & \textit{in } \Omega, \\ u = 0, & \textit{on } \partial\Omega, \end{cases} \tag{$S\mathscr{P}_p$}$$

where $\lambda > 0$. We study Problem ($S\mathscr{P}_p$) for a continuous potential $V : \overline{\Omega} \to \mathbb{R}$ such that $-\Delta + V$ is compact and positive definite[1]. *Under these assumptions, we work in $H_0^1(\Omega)$ with the scalar product $(u|v) = \int_\Omega \nabla u \cdot \nabla v + V(x)uv$ and associated norm*

$$\|u\|^2 = \int_\Omega |\nabla u|^2 + V(x)u^2. \tag{3.1}$$

The energy functional $\mathscr{E}_p$ is given by

$$H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 + V(x)u^2 - \frac{\lambda}{p}\int_\Omega |u|^p,$$

which does not change anything about the existence of ground state solutions (resp. l.e.n.s.). They can again be defined as minima of the energy functional $\mathscr{E}_p$ on the Nehari manifold

$$\mathscr{N}_p := \left\{ u \in H_0^1(\Omega) \setminus \{0\} \mid \int_\Omega |\nabla u|^2 + V(x)u^2 = \lambda \int_\Omega |u|^p \right\}$$

(resp. $\mathscr{M}_p := \{u : u^{\pm} \in \mathscr{N}_p\}$). Let us just remark that, if $-\Delta + V$ is not positive definite, the zero function may not be a local minimum of $\mathscr{E}_p$ anymore. There exist directions without functions belonging to $\mathscr{N}_p$. So, the approach defined in Chapter 1 does not work (see Proposition 1.1.1 or Chapter 6 for more information). Under our assumptions, we prove that our asymptotic behavior and symmetry results are still working. We denote by λ_i (resp. E_i) the distinct eigenvalues (resp. eigenspaces) of $-\Delta + V$ with DBC. As in Chapter 1, we show that a family of ground state solutions (resp. l.e.n.s.) converges up to a subsequence to a non-zero function if and only if $\lambda = \lambda_1$ (resp. λ_2). Depending on symmetries of eigenfunctions for $-\Delta + V$, we obtain the symmetry results. Of course, results given in Chapter 2 related to the $\mathscr{C}$-convergence can also be used in that case.

In Section 3.2, we study what happens for the q-Laplacian Lane–Emden Problem (QLEP)

$$\begin{cases} -\Delta_q u = \lambda |u|^{p-2}u, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{$Q\mathscr{P}_p$}$$

[1] The eigenvalues are positive.

Here are $\Delta_q u := \operatorname{div}(|\nabla u|^{q-2}\nabla u)$, $\lambda > 0$ *and* $1 < q < p < q^*$ *(with* $q^* = \frac{Nq}{N-q}$ *if* $q < N$, *and* $q^* = +\infty$ *otherwise). Let us just remark that, now, we need to deal with a nonlinear operator* Δ_q *and that the associated energy functional*

$$\mathscr{E}_{p,q} : W_0^{1,q}(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{q}\int_\Omega |\nabla u|^q - \frac{\lambda}{p}\int_\Omega |u|^p$$

is defined on a Banach space[2] $W_0^{1,q}(\Omega)$ *which is not a Hilbert space. Nevertheless, we obtain the existence of ground state solutions and least energy nodal solutions. The idea is similar as the previous linear case when* $q = 2$. *To be complete, it will be explained and proved in Section 3.2.1. Let us remark that energy functional* $\mathscr{E}_{p,q}$ *is a* $\mathscr{C}^2$*-functional for* $q \geq 2$ *and* $\mathscr{C}^1$*-functional for* $1 < q < 2$.

About asymptotic results, we prove that up to a rescaling, ground state solutions (resp. l.e.n.s.) of Problem ($Q\mathscr{P}_p$) *converge to non-zero solutions of the following nonlinear eigenvalue Problem* (3.2)

$$\begin{cases} -\Delta_q u = \lambda |u|^{q-2}u, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Unfortunately, we can say nothing about symmetries for now. Concerning the "implicit function theorem" method, we have some technical trouble to prove the bijectivity of the operators. Concerning the "general" method, we remark that we need the linearity of the left-hand side operator in Proposition 1.4.1. Moreover, in contrast to the linear case when $q = 2$, *even Problem* (3.2) *has not yet completely been solved. Let us denote by* λ_i *(resp.* E_i*) the eigenvalues (resp. eigenfunctions) of the q-Laplacian operator, i.e. solutions of Problem* (3.2)*. It is known (see e.g. [29]) that there exists a sequence of eigenvalues*

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots$$

with $\lambda_n \to +\infty$. *The first eigenvalue can be characterized as*

$$\lambda_1 := \inf_{u \in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}$$

[2] $W_0^{1,q}(\Omega)$ is the closure in $L^q(\Omega)$ of the space $\mathscr{C}_0^2(\Omega)$ for the classical norm $(\int_\Omega |\nabla u|^q)^{1/q}$.

and there exists only one eigenfunction e_1 (up to a multiplicative factor) which has constant sign (see e.g. [11]). The higher eigenvalues λ_n can be obtained through the following minimax principle: let us define the Krasnoselskii genus of a set $A \subseteq W_0^{1,q}(\Omega)$:

$$\gamma(A) := \min\left\{k \in \mathbb{N} \;\middle|\; \exists f : A \to \mathbb{R}^k \setminus \{0\},\ f \text{ continuous and odd}\right\}. \tag{3.3}$$

Define

$$\Gamma_k := \left\{A \subseteq W_0^{1,q}(\Omega) \;\middle|\; A \text{ symmetric, } A \cap \{||v||_q = 1\} \text{ compact, } \gamma(A) \geq k\right\}.$$

Then,

$$\lambda_k := \inf_{A \in \Gamma_k} \sup_{u \in A} \frac{\int_\Omega |\nabla u|^q}{\int_\Omega |u|^q}. \tag{3.4}$$

It turns out that there do not exist other eigenvalues between λ_1 and λ_2 (see e.g. [7]) but it is still an open question whether other eigenvalues can exist after λ_2. Eigenfunctions associated with higher eigenvalues must be sign-changing. Notice that if u and v are eigenfunctions with the identical eigenvalue, $u+v$ is in general not an eigenfunction, due to the nonlinearity of the problem. Due to elliptic regularity theory, eigenfunctions belong to $\mathscr{C}^{1,\alpha}_{loc}(\Omega)$, for $0 < \alpha < 1$ (see e.g. [26]).

During Section 3.2, by default, we denote by $\|.\|$ the norm in $W_0^{1,q}(\Omega)$.

The part related to the study of q-Laplacian is inspired by the paper [36] written in collaboration with E. Parini.

3.1 Positive definite Schrödinger problem

We study asymptotic behavior for a family $(u_p)_{p>2}$ of ground state solutions (resp. l.e.n.s.) for Problem ($S\mathscr{P}_p$).

3.1.1 Asymptotic behavior

Proposition 1.1.3 showing that the family $(u_p)_{p>2}$ is bounded still clearly holds. We just need to add $V(x)w$ in the Fredholm alternative and, of course, work with the norm 3.1. The lower bound proven in Proposition 1.1.5 is still working with similar modifications. With our assumptions on V, the norm (3.1) is equivalent to the classical norm $\left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}$. So, we can adapt Poincaré inequalities and Sobolev's embedding at norm (3.1). Thus, we obtain the same conclusion as for Theorem 1.1.8.

Theorem 3.1.1. *Let $(u_p)_{p>2}$ be a family of ground state solutions (resp. l.e.n.s.) of Problem ($S\mathscr{P}_p$). There exists $C > 0$ such that*

$$\|u_p\| \leqslant C\lambda^{\frac{1}{p-2}},$$

for $\lambda = \lambda_1$ (resp. λ_2). If $\lambda^{\frac{1}{2-p_n}} u_{p_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$ for some sequence $p_n \to 2$, then $\lambda^{\frac{1}{2-p_n}} u_{p_n} \to u_* \neq 0$ in $H_0^1(\Omega)$, u_* satisfies

$$\begin{cases} -\Delta u_*(x) + V(x)u_*(x) = \lambda u_*(x), & x \in \Omega, \\ u_* = 0, & x \in \partial\Omega, \end{cases}$$

and

$$\mathscr{E}_*(u_*) = \inf\{\mathscr{E}_*(u) : u \in E \setminus \{0\}, \langle \mathrm{d}\mathscr{E}_*(u), u \rangle = 0\},$$

where

$$\mathscr{E}_* : E \to \mathbb{R} : u \mapsto \frac{\lambda_2}{2} \int_\Omega u^2 - u^2 \log u^2$$

and $E = E_1$ (resp. E_2).

3.1.2 Symmetries extending

For this part, we are using the "general technique" presented in Section 1.4. Concerning the uniqueness result Lemma 1.4.1, with our assumptions on V, as previously, we just need to adapt Poincaré and Sobolev inequalities. Clearly, after this, Proposition 1.4.2 only depends on the nonlinearities. So, we obtain directly our abstract symmetry result. Of course, symmetries obtained are related to the structure of the eigenfunctions of $-\Delta + V$ and, thus, are related to the potential V.

Theorem 3.1.2. *Let $(G_\alpha)_{\alpha\in E}$ with $E = E_1$ (resp. E_2) be groups acting on $H_0^1(\Omega)$ in such a way that, for every $g \in G_\alpha$ and for every $u \in H_0^1(\Omega)$,*

$$g(E) = E, \quad g(E^\perp) = E^\perp, \quad g\alpha = \alpha \quad \text{and} \quad \mathscr{E}_p(gu) = \mathscr{E}_p(u).$$

Then, for all $M > 0$ and $\lambda = \lambda_1$ (resp. λ_2), if p is close enough to 2, any ground state (resp. least energy nodal) solution $u_p \in \{u \in B(0,M) : P_E(u) \notin B(0, \frac{1}{M})\}$ of Problem ($S\mathscr{P}_p$) belongs to the invariant set of G_{α_p} where $\alpha_p := P_E u_p$.

3.1.3 Symmetry breaking of least energy nodal solutions on rectangles

Concerning symmetry breaking, of course, it also depends on eigenfunctions of the operator $-\Delta + V$. If, on squares and rectangles, second eigenfunctions possess the same symmetries and if reduced functional $\mathscr{E}_*$ has the same structure as for the classical Lane-Emden Problem ($\mathscr{P}_p$), we also obtain the symmetry breaking.

Theorem 3.1.3. *If second eigenfunctions of $-\Delta + V$ respect the same symmetries as the second eigenfunctions of $-\Delta$ and if minima of $\mathscr{E}_*$ are not given by functions such that the nodal line is a median, there exist some rectangles such that least energy nodal solutions of Problem ($S\mathscr{P}_p$) are neither symmetric nor antisymmetric with respect to one of the medians of the rectangles.*

3.1.4 Example: non-zero constant potential

In this section, we illustrate the most natural example of a non-zero potential V: the case of a constant potential. Certainly, if we just consider a translation on the eigenvalues such that the first eigenvalue stays positive, $-\Delta + V$ stays positive definite and eigenfunctions are the same as for $-\Delta$.

By previous results, we obtain identical symmetries as for the Lane–Emden Problem (*LEP*). In particular, we have

- for p close to 2, the least energy nodal solution on a rectangle is even and odd with respect to a median;
- for p close to 2, the least energy nodal solution on radial domains are even with respect to $N-1$ orthogonal directions and odd with respect to the orthogonal one;
- for p close to 2, the least energy nodal solution on a square is odd with respect to the barycenter.

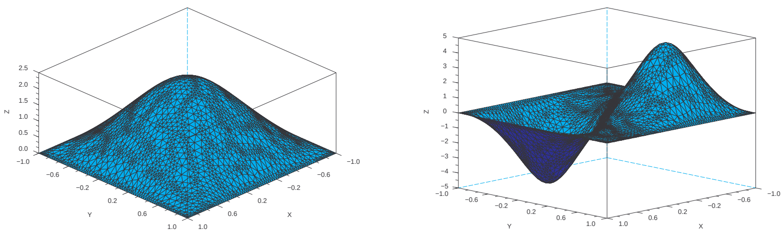


Figure 3.1: Numerical solutions in the case of a non-zero potential V.

In order to illustrate it, let us consider the following problem

$$\begin{cases} -\Delta u - \dfrac{\pi^2}{4} u = u^3, & \text{in } \Omega, \\ \qquad\quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

defined on the square $\Omega = (-1,1)^2$ in $\mathbb{R}^2$. First and second eigenvalues of $-\Delta$ are

$$\lambda_1 = \frac{\pi^2}{2} \quad \text{and} \quad \lambda_2 = \frac{5\pi^2}{4}.$$

Figure 3.1 and Table 3.1 show the results of the numerical experiments: one-signed (resp. nodal) numerical solutions have the expected symmetries. The nodal line of the sign-changing solution seems to be a diagonal.

	Initial function	$\min u$	$\max u$	$\mathscr{E}_4(u^+)$	$\mathscr{E}_4(u^-)$
$g.s.$	$(x-1)(y-1)(x+1)(y+1)$	0.0	2.31	2.99	0.0
$l.e.n.$	$\sin(\pi(x+1))\sin(2\pi(y+1))$	-4.6	4.6	16.6	16.6

Table 3.1: Characteristics of the approximate solutions for non-zero V.

Since the limit functional for this example is the same as the limit functional for the Lane–Emden Problem ($\mathscr{P}_p$), the symmetry breaking phenomenon occurs.

3.2 q-Laplacian Lane–Emden problem

3.2.1 Existence of solutions

Let us fix $1 < q < +\infty$. For the sake of completeness, we prove the existence of at least two non-trivial solutions to Problem ($Q\mathscr{P}_p$). Let us just remark that, as a byproduct, we prove the existence of ground state and least energy nodal solutions for all the other problems studied until now.

In order to do this, we work, for $q < p < q^*$, with the *Nehari manifold*

$$\mathscr{N}_p := \left\{ u \in W_0^{1,q}(\Omega) \setminus \{0\} \;\middle|\; \langle \mathrm{d}\mathscr{E}_{p,q}(u), u \rangle = \int_\Omega |\nabla u|^q - \lambda \int_\Omega |u|^p = 0 \right\}.$$

We will also make use of the *positive Nehari manifold*

$$\mathscr{N}_p^+ := \{u \in \mathscr{N}_p \,|\, u \geq 0\},$$

the *negative Nehari manifold*

$$\mathscr{N}_p^- := \{u \in \mathscr{N}_p \,|\, u \leq 0\}$$

and the *nodal Nehari set*

$$\mathscr{M}_p := \left\{ u \in \mathscr{N}_p \;\middle|\; u^+ \in \mathscr{N}_p^+, u^- \in \mathscr{N}_p^- \right\}.$$

The following results prove that ground state solutions are well-defined and characterized by functions minimizing the energy functional $\mathscr{E}_{p,q}$ in $\mathscr{N}_p$ and that least energy nodal solutions are well-defined and characterized by functions minimizing the energy functional $\mathscr{E}_{p,q}$ in $\mathscr{M}_p$.

Proposition 3.2.1. *For every $u \in W_0^{1,q}(\Omega) \setminus \{0\}$, there exists one and only one $t_p^* > 0$ such that $t_p^* u \in \mathscr{N}_p$. Moreover,*

$$\mathscr{E}_{p,q}(t_p^* u) = \max_{t>0} \mathscr{E}_{p,q}(tu).$$

Proof. For $u \in W_0^{1,q}(\Omega) \setminus \{0\}$, we have

$$tu \in \mathscr{N}_p \Leftrightarrow \int_\Omega |\nabla (tu)|^q - \lambda \int_\Omega |tu|^p = 0 \Leftrightarrow t^q \int_\Omega |\nabla u|^q - \lambda t^p \int_\Omega |u|^p = 0.$$

The last equation admits

$$t_p^* := \left(\frac{\int_\Omega |\nabla u|^q}{\lambda \int_\Omega |u|^p} \right)^{\frac{1}{p-q}} \tag{3.6}$$

as unique positive solution. For $t \geq 0$, we define

$$\psi(t) := \mathscr{E}_{p,q}(tu) = \frac{1}{q}\int_\Omega |\nabla(tu)|^q - \frac{\lambda}{p}\int_\Omega |tu|^p = \frac{t^q}{q}\int_\Omega |\nabla u|^q - \frac{\lambda t^p}{p}\int_\Omega |u|^p.$$

We have

$$\psi'(t) = t^{q-1}\int_\Omega |\nabla u|^q - \lambda t^{p-1}\int_\Omega |u|^p,$$

so that the only positive critical point is $t = t_p^*$. Since $\psi(0) = 0$ and $\psi(t) \to -\infty$ as $t \longrightarrow +\infty$, t_p^* must be a maximum point, which means

$$\mathscr{E}_{p,q}(t_u^*) = \max_{t>0} \mathscr{E}_{p,q}(tu). \quad \square$$

By the previous result and since the support of u^+ and u^- are disjoint, we obtain the following result.

Corollary 3.2.2. *For every $u \in W_0^{1,q}(\Omega) \setminus \{0\}$, the numbers $t_p^+, t_p^- > 0$ such that $t_p^+ u^+ + t_p^- u^- \in \mathscr{M}_p$ are uniquely defined.*

Proposition 3.2.3. *The Nehari manifold $\mathscr{N}_p$ is closed in $W_0^{1,q}(\Omega)$.*

Proof. Since $\mathscr{E}_{p,q}$ is of class $\mathscr{C}^1$, it is clear that $\mathscr{N}_p \cup \{0\}$ is closed. So we must prove that 0 is not an accumulation point for $\mathscr{N}_p$; this follows from the fact that the $W_0^{1,q}$-norm of every function $u \in \mathscr{N}_p$ is uniformly bounded from below. Indeed, from Sobolev's embedding theorem, we have

$$||\nabla v||_q \geq C||v||_p, \quad \forall\, v \in W_0^{1,q}(\Omega).$$

For $v \in W_0^{1,q}(\Omega) \setminus \{0\}$, the unique positive multiplicative function $t_p^* v \in \mathscr{N}_p$ (with t_p^* as in (3.6)) satisfies

$$\begin{aligned} ||\nabla(t_p^* v)||_q \geq C||t_p^* v||_p &= C\left(\frac{||\nabla v||_q^q}{\lambda ||v||_p^p}\right)^{\frac{1}{p-q}} ||v||_p \\ &= C\lambda^{\frac{-1}{p-q}} \left(\frac{||\nabla v||_q}{||v||_p}\right)^{\frac{q}{p-q}} \geq C^{\frac{p}{p-q}} \lambda^{\frac{-1}{p-q}}. \end{aligned}$$

$\square$

The following result proves that the minima of the energy functional $\mathscr{E}_{p,q}$ on the positive and negative Nehari manifold $\mathscr{N}_p$, and on the nodal Nehari set $\mathscr{M}_p$ exist.

Proposition 3.2.4. *The three infima*

$$\inf_{u\in\mathscr{N}_p^+} \mathscr{E}_{p,q}(u), \quad \inf_{u\in\mathscr{N}_p^-} \mathscr{E}_{p,q}(u), \quad \inf_{u\in\mathscr{M}_p} \mathscr{E}_{p,q}(u)$$

are attained.

Proof. We give a proof for $\mathscr{M}_p$. The arguments are simpler for $\mathscr{N}_p^+$ and $\mathscr{N}_p^-$. Let us define $c := \inf_{\mathscr{M}_p} \mathscr{E}_{p,q}$ and consider $(u_n)_{n\in\mathbb{N}} \subseteq \mathscr{M}_p$ such that $\mathscr{E}_{p,q}(u_n) \to c$.

Since $\mathscr{E}_{p,q}(v) = \left(\frac{1}{q} - \frac{1}{p}\right)\|v\|^q$ for any $v \in \mathscr{N}_p$, we obtain that $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,q}(\Omega)$. So, up to a subsequence, there exist u, v and w such that $u_n \rightharpoonup u$, $u_n^+ \rightharpoonup v$ and $u_n^- \rightharpoonup w$ in $W_0^{1,q}(\Omega)$. By Sobolev's embedding theorem and the continuity of the map $u \mapsto u^\pm$, we obtain that $u^+ = v$ and $u^- = w$.

By Proposition 3.2.3, we obtain that

$$\int_\Omega |u^+|^p = \lim_{n\to\infty} \int_\Omega |u_n^+|^p = \frac{1}{\lambda} \lim_{n\to\infty} \|u_n^+\|^q > 0.$$

So, u is a sign-changing function.

It remains to verify that $u \in \mathscr{M}_p$ and $u_n \to u$ in $W_0^{1,q}(\Omega)$. In fact, it suffices to prove that $u_n^+ \to u^+$ and $u_n^- \to u^-$ in $W_0^{1,q}(\Omega)$. Otherwise we can assume w.l.o.g. that u_n^+ does not converge to u^+. Then $\|u^+\|^q < \liminf_{n\to\infty}\|u_n^+\|^q$, which implies that $\langle \mathrm{d}\mathscr{E}_{p,q}(u^+), u^+\rangle < 0$. So u^+ does not belong to the Nehari manifold. By Proposition 3.2.1, there exist $0 < \alpha < 1$ and $0 < \beta \le 1$ such that $\alpha u^+ + \beta u^-$ belongs to $\mathscr{M}_p$. In fact, we have

$$\mathscr{E}_{p,q}(\alpha u^+ + \beta u^-) < \liminf_{n\to\infty}(\mathscr{E}_{p,q}(\alpha u_n^+) + \mathscr{E}_{p,q}(\beta u_n^-)) \le \liminf_{n\to\infty} \mathscr{E}_{p,q}(u_n) = c,$$

which is a contradiction. So the minimum of the energy on $\mathscr{M}_p$ is attained in u. □

The following results show that functions constructed in Proposition 3.2.4 are solutions of Problem ($Q\mathscr{P}_p$). Observe that, as the positive part and the negative part of a sign-changing solution belong to the Nehari manifold $\mathscr{N}_p$

and as the energy of the positive or negative part is strictly less than the energy of the solution, we obtain that the functions which minimize energy on the positive Nehari manifold $\mathscr{N}_p^+$ or negative Nehari manifold $\mathscr{N}_p^-$ are ground state solutions of Problem ($Q\mathscr{P}_p$). We will make use of the following lemma (see e.g. [4, 35] for a proof).

Lemma 3.2.5. *Let $B \subseteq \mathbb{R}^n$ be a convex set with piecewise $\mathscr{C}^1$boundary, let $f : \overline{B} \to \mathbb{R}^n$ be a continuous function. If f points inwards B on ∂B, then f possesses a zero in B.*

Theorem 3.2.6. *If the function $u_p \in \mathscr{M}_p$ (resp. $\mathscr{N}_p^+$ or $\mathscr{N}_p^-$) is such that $\mathscr{E}_{p,q}(u_p) = \inf_{u\in\mathscr{M}_p} \mathscr{E}_{p,q}(u)$ (resp. $\inf_{u\in\mathscr{N}_p^+} \mathscr{E}_{p,q}(u)$ or $\inf_{u\in\mathscr{N}_p^-} \mathscr{E}_{p,q}(u)$), then u_p is a critical point for $\mathscr{E}_{p,q}$.*

Proof. We give the proof for $\mathscr{M}_p$. The arguments are essentially the same for the two other cases. We only need to think that a minimum on $\mathscr{N}_p^+$ or $\mathscr{N}_p^-$ is a minimum on $\mathscr{N}_p$ which is a $\mathscr{C}^1$-manifold. So, for the two other cases, we do not need the deformation used in the next part of the proof.

Let us write $c := \min_{\mathscr{M}_p} \mathscr{E}_{p,q}$. Let us suppose that u_p is not a critical point for $\mathscr{E}_{p,q}$. Since $\mathscr{E}_{p,q}$ is of class $\mathscr{C}^1$, it is possible to find $\varepsilon > 0$ and a ball B centered at u_p such that

$$c - \varepsilon \leq \mathscr{E}_{p,q}(u) \leq c + \varepsilon, \quad \forall u \in B,$$

and

$$\| \mathrm{d}\mathscr{E}_{p,q}(u)\|_{(W_0^{1,q})'} \geq \varepsilon, \quad \forall u \in B.$$

Let us consider the quarter of a plane π defined as

$$\pi := \{\alpha u_p^+ + \beta u_p^- \mid \alpha, \beta > 0\}.$$

Notice that, from Proposition 3.2.1, u_p is the unique global maximum of $\mathscr{E}_{p,q}$ on π. We have $\sup_\pi \mathscr{E}_{p,q} = c$. By the deformation lemma (see e.g. [30]), there exists a deformation $\Gamma : [0,1] \times \overline{B} \cap \pi \to W_0^{1,q}(\Omega)$ such that

1. $\Gamma(0,u) = u$ for $u \in B \cap \pi$,
2. $\mathscr{E}_{p,q}(\Gamma(t,u)) < c$ for $u \in B \cap \pi$ and $t \in (0,1)$,

3. $\Gamma(t,u) = u$ for $u \in \partial B \cap \pi$ and $t \in [0,1]$, and

4. $||\Gamma(t,u) - u|| \leq 8t$ for $u \in B \cap \pi$ and $t \in [0,1]$.

Because of the compactness of $\overline{B} \cap \pi$, it is possible to find $t^* > 0$ such that $\Gamma(t^*,u)$ is a sign-changing function for every $u \in B \cap \pi$.

Now, we consider the application

$$\psi : \pi \to \mathbb{R}^2 : v \mapsto \left(\langle \mathrm{d}\mathscr{E}_{p,q}(\Gamma(t^*,v)^+), \Gamma(t^*,v)^+\rangle, \langle \mathrm{d}\mathscr{E}_{p,q}(\Gamma(t^*,v)^-), \Gamma(t^*,v)^-\rangle\right).$$

Since $\Gamma(t^*,v) = v$ on ∂B, we obtain that the vector field points inwards on ∂B (see Figure 3.2). Using Lemma 3.2.5, we obtain that there exists $w \in B \cap \pi$ such that $\Gamma(t^*,w) \in \mathscr{M}_p$. This is a contradiction because $\mathscr{E}_{p,q}(\Gamma(t^*,w)) < c$. $\square$

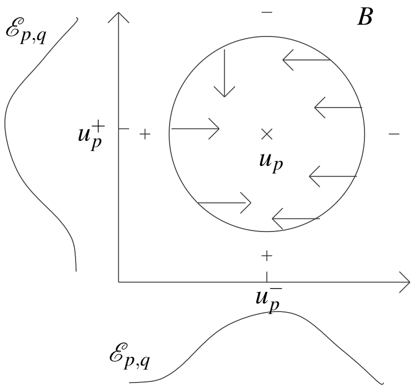


Figure 3.2: Use of Lemma 3.2.5.

3.2.2 Asymptotic results

In this section, we study the asymptotic behavior of ground state solutions u_p (resp. l.e.n.s.) of Problem ($\mathcal{QP}_p$) when p goes to q. We prove that $\lambda = \lambda_1$ (resp. λ_2) is the "good" rescaling factor to study convergence of family $(u_p)_{p>q}$ of ground state solutions (resp. l.e.n.s.). As in Chapter 1, "good" rescaling means that we have the following results.

Theorem 3.2.7. *As $p \to q$, ground state solutions u_p of Problem ($Q\mathscr{P}_p$)*
(i) diverge to infinity if $\lambda < \lambda_1$;
(ii) converge to a non-zero first eigenfunction of the q-Laplacian, up to a subsequence, if $\lambda = \lambda_1$;
(iii) converge to zero if $\lambda > \lambda_1$.

Theorem 3.2.8. *As $p \to q$, least energy nodal solutions u_p of Problem ($Q\mathscr{P}_p$)*
(i) diverge to infinity if $\lambda < \lambda_2$;
(ii) converge to a non-zero second eigenfunction of the q-Laplacian, up to a subsequence, if $\lambda = \lambda_2$;
(iii) converge to zero if $\lambda > \lambda_2$.

In fact, we have that there exist suitable positive constants C_1 and C_2 such that

$$C_1 \left(\frac{\lambda_1}{\lambda}\right)^{\frac{1}{p-q}} \leq \|u_p\| \leq C_2 \left(\frac{\lambda_1}{\lambda}\right)^{\frac{1}{p-q}}$$

if u_p is a ground state solution, and

$$C_1 \left(\frac{\lambda_2}{\lambda}\right)^{\frac{1}{p-q}} \leq \|u_p\| \leq C_2 \left(\frac{\lambda_2}{\lambda}\right)^{\frac{1}{p-q}}$$

if u_p is a least energy nodal solution.

Let us mention that the case $\lambda < \lambda_1$ in Theorem 3.2.7 was already investigated in [42].

Let us first remark that statements (i) and (iii) of Theorems 3.2.7 and 3.2.8 can directly be derived from (ii) as follows. If v_p is a ground state solution of Problem ($Q\mathscr{P}_p$) with $\lambda = \lambda_1$, then $u_p := \left(\frac{\lambda_1}{\mu}\right)^{\frac{1}{p-q}} v_p$ will be a ground state solution for $\lambda = \mu$. So for $\lambda < \lambda_1$, as $p > q$ and $p \to q$, $u_p := \left(\frac{\lambda_1}{\lambda}\right)^{\frac{1}{p-q}} v_p$ goes to infinity, while for $\lambda > \lambda_1$ it goes to zero. The proof of Theorem 3.2.8 (i) and (iii) is virtually identical. It remains to study (ii). We consider the case $\lambda = \lambda_1$ (resp. λ_2) for ground state (resp. least energy nodal) solutions. The family $(u_{p,1})_{p>q}$ will denote a family of ground state solutions for the problem ($Q\mathscr{P}_p$) with $\lambda = \lambda_1$, while $(u_{p,2})_{p>q}$ will be a family of least energy nodal solutions for the same problem with $\lambda = \lambda_2$.

The process is more or less the same as in Chapter 1. In the proofs, we just point out the differences with the linear case when $q = 2$. Let us fix a first non-zero eigenfunction e_1 and a non-zero second eigenfunction e_2 of $-\Delta_q$.

Lemma 3.2.9. *For $q < r < q^*$, the quantities $\sup_{q<p<r} t_p^*$, $\sup_{q<p<r} t_p^+$ and $\sup_{q<p<r} t_p^-$ are finite, where t_p^*, t_p^+ and t_p^- are the unique positive real numbers such that $t_p^* e_1 \in \mathcal{N}_p$ and $t_p^+ e_2^+ + t_p^- e_2^- \in \mathcal{M}_p$.*

Proof. The proof is exactly the same as for Lemma 1.1.1. □

Proposition 3.2.10. *The families $(u_{p,1})_{p>q}$ and $(u_{p,2})_{p>q}$ are uniformly bounded in $W_0^{1,q}(\Omega)$.*

Proof. Again, the proof is the same as for Proposition 1.1.2. □

About Proposition 1.1.3, as we need the Fredholm alternative, we cannot work in the same way. The two following results prove that the sequence of ground state solutions (resp. least energy nodal solutions) of Problem ($Q\mathscr{P}_p$) stays away from zero.

Proposition 3.2.11. *Let $(u_{q,1})_{q>p}$ be a family of ground state solutions of Problem ($Q\mathscr{P}_p$) for $\lambda = \lambda_1$. Then*

$$\liminf_{p\to q} ||\nabla u_{p,1}||_q > 0.$$

Proof. We are working in the same vein as Proposition 1.1.6. In fact, the study of ground state solutions simplifies the arguments. We do not need to use Lemma 1.1.5 and we directly can use the Hölder inequality on the ground state solutions. □

Concerning least energy nodal solutions, we cannot use the Poincaré inequality as in Proposition 1.1.6. To solve this problem, we use the Krasnoselskii genus 3.3 defined in the Introduction of this chapter.

Proposition 3.2.12. *Let $(u_{p,2})_{p>q}$ be a family of least energy nodal solutions of Problem ($Q\mathscr{P}_p$) for $\lambda = \lambda_2$. Then*

$$\liminf_{p\to q} ||\nabla u_{p,2}||_q > 0.$$

Proof. Since $u_{p,2}$ is sign-changing, we can write $u_{p,2} = u^+_{p,2} + u^-_{p,2}$, with $u^+_{p,2}, u^-_{p,2} \neq 0$. Define

$$A := \left\{ v \in W_0^{1,q}(\Omega) \setminus \{0\} \mid v = \alpha u^+_{p,2} + \beta u^-_{p,2},\ (\alpha, \beta) \neq (0,0) \right\}.$$

As consequence of the intermediate value theorem, it can be proved that $A \in \Gamma_2$ as defined in the Krasnoselskii genus defined by (3.3). Hence, by definition of λ_2, we have

$$\begin{aligned}
\lambda_2 &\leq \max_{(\alpha,\beta)\neq(0,0)} \frac{|\alpha|^q ||\nabla u^+_{p,2}||_q^q + |\beta|^q ||\nabla u^-_{p,2}||_q^q}{|\alpha|^q ||u^+_{p,2}||_q^q + |\beta|^q ||u^-_{p,2}||_q^q} \\
&\leq \max \left\{ \frac{||\nabla u^+_{p,2}||_q^q}{||u^+_{p,2}||_q^q}, \frac{||\nabla u^-_{p,2}||_q^q}{||u^-_{p,2}||_q^q} \right\}.
\end{aligned}$$

The last inequality follows from the fact that

$$\frac{a+b}{c+d} \leq \frac{a}{c} \iff \frac{b}{d} \leq \frac{a}{c} \qquad \text{for any } a, b, c, d > 0.$$

Let us assume, without loss of generality, that the maximum is attained for $u^+_{p,2}$. Then we have

$$\lambda_2 ||u^+_{p,2}||_q^q \leq ||\nabla u^+_{p,2}||_q^q.$$

Fix $r > 0$ such that $q \leq p < r < q^*$ and set $s := \frac{r(p-q)}{p(r-q)}$. By Hölder's inequality, we obtain, on one hand,

$$||u^+_{p,2}||_p^q \leq ||u^+_{p,2}||_q^{q-qs} ||u^+_{p,2}||_r^{qs}.$$

On the other hand, since $(u^+_{p,2})_{p>q}$ belongs to the Nehari manifold $\mathcal{N}_p$, we have

$$||\nabla u^+_{p,2}||_q^q = \lambda_2 ||u^+_{p,2}||_p^p$$

and since $r < q^*$ by Sobolev's embedding Theorem we get

$$||u^+_{p,2}||_r^q \leq C ||\nabla u^+_{p,2}||_q^q.$$

So, it follows that

$$||\nabla u^+_{p,2}||_q \geq \lambda_2^{\frac{-q+p-sp}{pq-q^2}} C^{-\frac{sp}{pq-q^2}}$$

and, putting in the value of s,

$$||\nabla u_{p,2}^+||_q \geq \lambda_2^{\frac{1}{q-r}} C^{\frac{r}{q(q-r)}}.$$

From the relation

$$||\nabla u_{p,2}||_q \geq ||\nabla u_{p,2}^+||_q$$

and since the estimate does not depend on p, we obtain the claim. □

Theorem 3.2.13. *Let $(u_{p,1})_{p>q}$ be a family of ground state solutions of Problem ($Q\mathscr{P}_p$) for $\lambda = \lambda_1$ (resp. $(u_{p,2})_{p>q}$ be a family of least energy nodal solutions for $\lambda = \lambda_2$). Then, up to a subsequence, $u_{p,1} \to u_*$ (resp. $u_{p,2} \to u_*$) in $L^q(\Omega)$ as $p \to q$, where the function u_* is a non-zero first (resp. second) eigenfunction of the q-Laplacian.*

Proof. We give the proof for the family of least energy nodal solutions. The idea is the same for the family of ground state solutions. Let $v \in W_0^{1,q}(\Omega)$. Because of the uniform boundedness of the $(u_{p,2})_{p>q}$ in $W_0^{1,q}(\Omega)$, there exists $u_* \in W_0^{1,q}(\Omega)$ such that $u_{p,2} \rightharpoonup u_*$ in $W_0^{1,q}(\Omega)$ and $u_{p,2} \to u_*$ in $L^q(\Omega)$ for $p \to q$ (up to a subsequence). By Lebesgue's dominated convergence theorem we also have

$$|u_{p,2}|^{p-2}u_{p,2} \to |u_*|^{q-2}u_* \text{ in } L^q(\Omega).$$

So,

$$\begin{aligned}
\int_\Omega |\nabla u_*|^{q-2}\nabla u_* \nabla v &= \lim_{p\to q} \int_\Omega |\nabla u_{p,2}|^{q-2}\nabla u_{p,2}\nabla v \\
&= \lim_{p\to q} \lambda_2 \int_\Omega |u_{p,2}|^{p-2}u_{p,2}v \\
&= \lambda_2 \int_\Omega |u_*|^{q-2}u_* v.
\end{aligned}$$

By Theorem 3.2.12, $u_* \neq 0$. Hence, u_* is a non-zero second eigenfunction of $-\Delta_q$. □

Chapter 4

General nonlinearities

In this chapter, we consider some perturbations of $f_p(x) = |x|^{p-2}x$, the nonlinearity studied for the Lane–Emden Problem ($\mathscr{P}_p$) with DBC in Chapter 1. We work with Problem ($G\mathscr{P}_p$)

$$\begin{cases} -\Delta u = f_p(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$G\mathscr{P}_p$}$$

where Ω is an open bounded connected domain of $\mathbb{R}^N$, $N \geqslant 2$, $2 < p < 2^*$, $f_p(0) = 0$ and $f_p \in \mathscr{C}^1(\mathbb{R})$. We are interested in non-necessarily homogeneous nonlinearities and we would like to know whether results given in Chapter 1 are still valid or not.

Under some growth assumptions, the solutions of Problem ($G\mathscr{P}_p$) are the critical points of the energy functional

$$\mathscr{E}_p : H_0^1(\Omega) \to \mathbb{R} : u \to \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F_p(u),$$

where $F_p(t) := \int_0^t f_p(s)\,\mathrm{d}s$. To explain this, let us denote $M(\mathbb{R})$ the set of Borel-measurable functions from $\mathbb{R}$ to $\mathbb{R}$. For a function $h \in M(\mathbb{R})$ and positive reals q and γ, we refer respectively to a polynomial or an exponential growth condition by

(PG_q) *there exists $C > 0, \forall t \in \mathbb{R}, |h(t)| \leqslant C(|t|^q + 1)$;*

(EG_γ) *there exists* $C > 0, \forall t \in \mathbb{R}, |h(t)| \leqslant Ce^{\gamma t^2}$.

The classical subcritical and exponential growth assumptions are defined as

(GA_1) *for all* $p > 2, \exists\, q \in (2, 2^*)$ *such that* f_p *satisfies* (PG_{q-1})*;*

(GA_1') *for all* $p > 2, \exists 0 < \gamma < \gamma_0$ *such that* f_p *satisfies* (EG_γ).

The condition (GA_1) *is motivated by the classical Sobolev embedding theorem while the latter is related to the Trudinger-Moser inequality when* $N = 2$*: for all* $\gamma > 0$ *and all* $u \in H^1(\Omega)$, $\int_\Omega e^{\gamma u^2} < +\infty$ *and, moreover, there exists* $\gamma_0 = \gamma_0(\Omega) > 0$ *such that*

$$\sup\left\{ \int_\Omega e^{\gamma u^2} : \|u\| \leqslant 1 \right\} < +\infty$$

if and only if $\gamma < \gamma_0$ *[52]. These assumptions ensure that the energy functional* $\mathscr{E}_p$ *is well-defined and of class* $\mathscr{C}^1(\Omega)$ *on* $H_0^1(\Omega)$ *(see e.g. [35]). Let us also note that in the case of assumption* (GA_1') *in dimension* 2*, the following lemma, proved in [15], plays the role of the standard compactness argument when assuming a subcritical polynomial growth.*

Lemma 4.0.14. *Let* $u_n, v_n \in H^1(\Omega)$ *such that* $\|u_n\|, \|v_n\| \leqslant 1$*. If* $u_n \rightharpoonup u, v_n \rightharpoonup v$ *in* $H^1(\Omega)$ *and* $\gamma_n \to \gamma < \gamma_0$*, then, for every* $p \in [1, +\infty)$ *and every* $q \in [1, \frac{\gamma_0}{\gamma})$, $v_n^p e^{\gamma_n u_n^2} \to v^p e^{\gamma u^2}$ *in* $L^q(\Omega)$.

To obtain the existence of non-zero solutions, we consider the following standard assumption (GA_2) *giving a mountain pass structure (see Figure 1.1 in Chapter* 1*) to the functional* $\mathscr{E}_p$ *with a Nehari fibering method.*

(GA_2) *(a) For all* $p > 2, \lim_{t\to 0} \frac{f_p(t)}{|t|} = 0$*;*

(b) for all $p > 2, \lim_{|t|\to\infty} \frac{f_p(t)}{t} = +\infty$*;*

(c) for all $p > 2, \forall t \neq 0, \left(\frac{f_p(t)}{|t|}\right)' > 0$.

As in Chapter 1*, we define the Nehari manifold* $\mathscr{N}_p$ *(resp. the nodal Nehari set* $\mathscr{M}_p$*) as*

$$\mathscr{N}_p := \left\{ u \in H_0^1(\Omega) \setminus \{0\} \mid \|u\|^2 = \int_\Omega f_p(u)u \right\}$$

(resp.

$$\mathscr{M}_p := \{u \in H_0^1(\Omega) : u^\pm \in \mathscr{N}_p\}).$$

Under the assumptions $(GA_1)-(GA_2)$, J. Van Schaftingen and M. Willem obtained in [68] the existence of ground state and least energy nodal solutions. Ground state (resp. least energy nodal) solutions are the critical points of $\mathscr{E}_p$ with minimum energy on $\mathscr{N}_p$ (resp. $\mathscr{M}_p$). They generalized the existence result of A. Castro, J. Cossio and J. M. Neuberger where additional assumptions (like the super-quadraticity (AA_1) here under) were required (see e.g. [23] or Proposition 3.2.6).

We will always suppose that either assumptions $(GA_1)-(GA_2)$ hold or $N = 2$ and $(GA_1')-(GA_2)$ hold.

To obtain expected asymptotic behavior of a family of ground state solutions (resp. l.e.n.s.) $(u_p)_{p>2}$, see Section 1.1, we work in Section 4.1 with the classical super-quadraticity assumption (AA_1).

(AA_1) *For all $p > 2, \forall t \in \mathbb{R}, f_p(t)t \geqslant pF_p(t)$.*

In some sense, we are working with a family of problems parametrized by super-quadraticity constants. Assumption (AA_1) implies that $|F_p|$ is comparable to $|t|^p$ and ensures that if $u \in \mathscr{N}_p$, then

$$\mathscr{E}_p(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F_p(u) \geqslant \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_\Omega f_p(u)u = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2. \quad (4.1)$$

Moreover, we need the following assumption (AA_2). Like usual, λ_1 (resp. λ_2) defined in $(AA_2)(a)$ will be used for ground state solutions (resp. l.e.n.s.).

(AA_2)
- *(a) For all $t \in \mathbb{R}, \lim_{p\to 2} f_p(t) =: f_2(t) = \lambda t$, for some $\lambda \in \{\lambda_1, \lambda_2\}$, and, for all $t_0 \in \mathbb{R}$, $\lim_{(p,t)\to(2,t_0)} \frac{f_p(t)-f_2(t)}{p-2} =: f_*(t_0) \in \mathbb{R}$;*
- *(b) if f_p satisfies (GA_1) (resp. (GA_1')), there exist $q \in (2, 2^*)$ (resp. $\gamma > 0$) and $h \in M(\mathbb{R})$ verifying (PG_q) (resp. (EG_γ)) such that, for any $p > 2, \forall t \in \mathbb{R}, |f_p(t)| \leqslant h(t)$;*
- *(c) there exist $q \in [2, 2^*]$ (resp. $\gamma > 0$) and $h \in M(\mathbb{R})$ verifying (PG_{q-1}) (resp. (EG_γ)) such that $\left|\frac{f_p(t)-f_2(t)}{p-2}\right| \leqslant h(t)$, for all p close to 2 and $t \in \mathbb{R}$;*
- *(d) (i) $\lim_{t\to 0} \frac{f_*(t)}{t} < 0$, (ii) $\lim_{|t|\to\infty} \frac{f_*(t)}{t} > 0$, (iii) $t \mapsto \frac{f_*(t)}{|t|}$ increasing.*

Roughly, it mainly means

$$f_p(t) = \lambda t + f_*(t)(p-2) + g_p(t), \tag{4.2}$$

where $g_p(t)$ behaves like $o(p-2)$. These assumptions permit us to define the reduced functional

$$\mathscr{E}_* : E \to \mathbb{R} : u \mapsto -\int_\Omega F_*(u), \text{ where } F_*(t) := \int_0^t f_*(s)\,\mathrm{d}s,$$

and E is the eigenspace of $-\Delta$ with DBC related to λ. We also define the associated reduced Nehari manifold

$$\mathscr{N}_* := \{u \in E \setminus \{0\} : \langle \mathrm{d}\mathscr{E}_*(u), u\rangle = 0\}.$$

One can easily check that $\mathscr{E}_*$ is well-defined and does have a mountain pass geometry. These assumptions also imply that any weak accumulation point u_* of $(u_p)_{p>2}$ is a critical point of the functional $\mathscr{E}_*$. We obtain the boundedness of $(u_p)_{p>2}$ under the assumptions $(AA_1)-(AA_3)$.

(AA_3) *(a) For all $t_0 \in \mathbb{R}, \lim_{(p,t)\to(2,t_0)} \frac{f_p'(t)t^2 - f_p(t)t}{p-2} =: H_*(t_0)$;*

(b) there exist $q \in [2, 2^*]$ (resp. $\gamma > 0$) and $h \in M(\mathbb{R})$ verifying (PG_{q-1}) (resp. (EG_γ)) such that,

$$\forall p > 2,\ \forall t \in \mathbb{R}, \quad \left|\frac{f_p'(t)t - f_p(t)}{p-2}\right| \leqslant h(t);$$

(c) there exists a critical point u_* of $\mathscr{E}_*$ such that $\int_\Omega H_*(u_*) \neq 0$.

Roughly, it mainly means that u_* is an nondegenerate critical point of $\mathscr{E}_*$. Let us also note that $(AA_2)(a)$ and $(AA_3)(a)$ can be rephrased as a convergence in p locally uniform with respect to t.

If u_* is a weak accumulation point of $(u_p)_{p>2}$, then it is different from 0 under the assumption (AA_2) and (AA_4) (resp. (AA_4')). Let us remark that we do not need anymore the assumption (AA_3) (resp. (AA_2) and (AA_3)).

(AA_4) *There exists $\eta > 0$ s.t., $\forall p < 2+\eta$ and $\forall t \in (-\eta, \eta) \setminus \{0\}$, $\frac{f_p(t)}{t} < \lambda$. Moreover, Ω is a domain of class $\mathscr{C}^2$ (or a product of intervals).*

(AA_4') *There exists $q \in (2, 2^*), \forall p > 2, \exists c_1(p), c_2(p) > 0$,*

$$\forall t \in \mathbb{R},\ |f_p(t)| \leqslant \lambda |t|^{1+c_1(p)} + c_2(p)|t|^{q-1},$$

with $\lim_{p\to 2} c_i(p) = 0$ *and* $\exists k \in \mathbb{R}, \forall p > 2, \frac{\log(1-c_2(p))}{c_1(p)} \geqslant k$,

This assumption (AA_4) is the uniform version of assumption $(GA_2)(a)$. In some sense, assumption (AA_4) is more general than assumption (AA_4') but requires some smooth conditions on $\partial\Omega$ to obtain a $\mathscr{C}$-convergence of u_p (see Remark 2.1.5).

To obtain the expected symmetries (see Section 1.4), we give in Section 4.2 assumptions implying, in any ball of $H_0^1(\Omega)$, the uniqueness of solutions up to the projection on some eigenspace E. Let us remark that we are more general than in Chapter 1 in the sense that solutions are not necessarily ground state or least energy nodal solutions and the eigenfunctions are not necessarily the first or second ones. To do it, we consider assumptions (SA_1) and (SA_2) (or (SA_2')). Assumption (SA_1) replaces some parts of the assumption (AA_2).

(SA_1) *(a) For all $t_0 \in \mathbb{R} : \lim_{p\to 2, t\to t_0} f_p(t) =: f_2(t_0) = \lambda t_0$, for some eigenvalue λ of $-\Delta$ in $H_0^1(\Omega)$;*

(b) if f_p satisfies (GA_1) (resp. (GA_1')), there exist $q \in (2, 2^*)$ (resp. $\gamma > 0$) and $h \in M(\mathbb{R})$ verifying (PG_q) (resp. (EG_γ)) such that, $\forall p > 2, \forall t \in \mathbb{R}, |f_p(t)| \leqslant h(t)$.

(SA_2) *(a) For all $u_0 \neq 0$, $\lim_{(s,t,p)\to(u_o,u_o,2)} \frac{f_p(s)-f_p(t)}{t-s} = \lambda$;*

(b) there exist $q \in [2, 2^*]$ and $h_a, h_b \in M(\mathbb{R})$ verifying (PG_{q-2}) such that, for every $p > 2$ and all $s \neq t \in \mathbb{R}$,

$$\left|\frac{f_p(t)-f_p(s)}{t-s}\right| \leqslant h_a(s) + h_b(t).$$

(SA_2') *(a) For all $u_0 \neq 0$, $\lim_{(s,t,p)\to(u_o,u_o,2)} \frac{f_p(s)-f_p(t)}{t-s} = \lambda$;*

(b) there exist $\gamma > 0$ and $h_a, h_b \in M(\mathbb{R})$ verifying (EG_γ) such that, for every $p > 2$ and all $s \neq t \in \mathbb{R}$,

$$\left|\frac{f_p(t)-f_p(s)}{t-s}\right| \leqslant h_a(s) + h_b(t).$$

In assumption $(SA_1)(a)$, we suppose that the limit of $f_p(t)$ is λt, with λ being precisely an eigenvalue (not necessarily λ_1 or λ_2). When looking at the Lane–Emden equation $(\mathscr{P}_p)$, one easily understands that this is not restrictive (see

Theorems 3.2.7 and 3.2.8). At least heuristically, one can see that if $\lambda \neq \lambda_i$, any converging sequence of solutions goes to either zero or infinity. That is why we have systematically studied the renormalized equation. Let us just remark that, with general nonlinearities, a changement of λ could change symmetries of solutions. Under (SA_1) and (SA_2), we obtain that if a bounded family of solutions $(u_p)_{p>2}$ is staying away from zero, for p close enough to 2, u_p respects the symmetries of its projection on the eigenspace E associated with λ.

As for the Lane–Emden Problem ($\mathscr{P}_p$) with a homogeneous nonlinearity (see Section 1.5), it is worth pointing out that symmetry results are not uniform with respect to the domain. Namely, we are able to build rectangles close to the square where least energy nodal solutions are neither symmetric nor antisymmetric with respect to the medians of the rectangles.

Let us just note that there exist non homogeneous nonlinearities respecting all the previous assumptions. We will analyze in Section 4.3 the three following cases for f_p:

$$\lambda t|t|^{p-2} + (p-2)t|t|^{q-2}, \quad \lambda t(e^{t^2} - 1)^{p-2} \quad \text{and } \lambda t \left(\sum_{i=1}^{k} \alpha_i |t|^{\beta_i(p)} \right),$$

i.e. the case of a subcritical superlinear ($2 < q < 2^*$) perturbation of a slowly growing homogeneous superlinearity, of a slowly exponentially growing subcritical nonlinearity in dimension two, and of a sum of small powers. These cases will be numerically illustrated.

To finish, in Section 4.4, we provide another "geometrical" assumption replacing (AA_1) and (AA_3) and implying the boundedness of the families of ground state solutions and least energy nodal solutions in $H_0^1(\Omega)$. We basically introduce a super-lower homogeneity assumption.

This work is inspired by the paper [12] written in collaboration with D. Bonheure and V. Bouchez.

4.1 Asymptotic behavior

In this section, we work out a priori estimates for ground state and least energy nodal solutions, and we analyze their asymptotic behavior when $p \to 2$. In the sequel of this section, λ exclusively takes the value λ_1, when we discuss ground state solutions, and λ_2 when we deal with least energy nodal solutions. The space E will denote the eigenspace of $-\Delta$ with DBC related to λ.

4.1.1 Limit equation

We turn our attention to the candidate accumulation points of $(u_p)_{p>2}$, as $p \to 2$. The following result generalizes Lemma 1.1.7.

Lemma 4.1.1. *Let $(u_p)_{p>2}$ be a bounded family of ground state (resp. least energy nodal) solutions of Problem $(G\mathscr{P}_p)$. If $p_n \to 2$ and $u_{p_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$, then, under assumptions $(GA_1)-(GA_2)$ and $(AA_2)-(AA_3)$, $u_{p_n} \to u_*$ in $H_0^1(\Omega)$ and u_* solves*

$$\begin{cases} -\Delta u_* = \lambda u_*, & \text{in } \Omega, \\ u_* = 0, & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega f_*(u_*)v = 0, & \forall v \in E. \end{cases}$$

Proof. The proof is similar as Lemma 1.1.7. By assumption (AA_2), we obtain that

$$f_{p_n}(u_{p_n}) \to f_2(u_*) \text{ in } L^2(\Omega), \text{ so that } \int_\Omega \nabla u_* \nabla v = \lambda \int_\Omega u_* v.$$

Then, by using Lebesgue's dominated convergence theorem and (AA_3), we obtain

$$0 = \lim_{n\to\infty} \int_\Omega \frac{(f_{p_n}(u_{p_n}) - \lambda u_{p_n})v}{p_n - 2} = \lambda \int_\Omega f_*(u_*)v.$$

About the convergence in $H_0^1(\Omega)$, we have

$$\begin{aligned} \int_\Omega |\nabla u_p - \nabla u_*|^2 &= \int_\Omega |\nabla u_p|^2 - 2\int_\Omega \nabla u_p \nabla u_* + \int_\Omega |\nabla u_*|^2 \\ &= \int_\Omega f_p(u_p)u_p - 2\int_\Omega \nabla u_p \nabla u_* + \lambda \int_\Omega u_*^2 \end{aligned}$$

which converges to $\lambda \int_\Omega u_*^2 - 2\int_\Omega \nabla u_* \nabla u_* + \lambda \int_\Omega u_*^2 = 0$. □

4.1.2 Upper bound

To obtain an upper bound, we construct appropriate test functions leading to an estimate of the energy and the norm of ground state solutions (resp. l.e.n.s.) of

Problem $(G\mathscr{P}_p)$ (as in Proposition 1.1.3). Let us consider a family $(v_p)_{p>2} \in H_0^1(\Omega) \setminus \{0\}$ and

$$h_p : (0, +\infty) \to \mathbb{R} : t \mapsto \int_\Omega t^{-1} f_p(tv_p)v_p.$$

This last function is well-defined because of assumption (GA_2). Moreover, h_p is continuous, increasing and $\lim_{t\to+\infty} h_p(t) = +\infty$. Let us denote by h_p^{-1} the inverse of h_p. In the next statement, we denote by $\hat{v}_p$ the classical projection of v_p on $\mathscr{N}_p$ along rays, which is characterized by

$$\hat{v}_p = t_p v_p, \quad \text{where } h_p(t_p) = \|v_p\|^2.$$

The next lemma generalizes Proposition 1.1.3. To understand well where the assumptions are playing a role, we give the entire proof.

Lemma 4.1.2. *Let $v_p := v_* + (p-2)w$, where w is the unique function in $E^\perp$ verifying*

$$\begin{cases} -\Delta w - \lambda w = f_*(v_*), & \text{in } \Omega, \\ \qquad\quad w = 0, & \text{on } \partial\Omega, \end{cases}$$

and v_* is a critical point of $\mathscr{E}_*$ such that $\int_\Omega H_*(v_*) \neq 0$. Then, under assumptions $(GA_1) - (GA_2)$ and $(AA_2) - (AA_3)$, the family $(\hat{v}_p)_{p>2}$ converges to v_*.

Proof. We focus on the case $\lambda = \lambda_1$. We need to prove that $t_p \to 1$. By the Fredholm alternative, w is well-defined. We then compute

$$\begin{aligned} \|v_p\|^2 &= \int_\Omega (\lambda v_* + \lambda(p-2)w + (p-2)f_*(v_*))v_p \\ &= \int_\Omega (\lambda v_p + (p-2)f_*(v_*))v_p \\ &= \int_\Omega f_p(v_p)v_p + (p-2)\left(\int_\Omega \frac{\lambda v_p - f_p(v_p)}{p-2} v_p + \int_\Omega f_*(v_*)v_p\right) \\ &= \int_\Omega f_p(v_p)v_p + (p-2)m_p, \end{aligned}$$

where, by (AA_2), $m_p \to 0$, as $p \to 2$. We therefore deduce, thanks to (GA_2), that

$$t_p = h_p^{-1}\left(\int_\Omega f_p(v_p)v_p + (p-2)m_p\right).$$

By defining $k_p := \int_\Omega f_p(v_p)v_p$, we have $h_p^{-1}(k_p) = 1$. So, we need to prove that

$$\lim_{p\to 2} h_p^{-1}(k_p + m_p(p-2)) - h_p^{-1}(k_p) = 0.$$

It is true since

$$\mathrm{d}_t h_p(t) = \int_\Omega t^{-1} f_p'(tv_p)v_p^2 - t^{-2} f_p(tv_p)v_p \tag{4.3}$$

and $(p-2)(h_p^{-1})'$ is uniformly bounded (with respect to p) in a neighbourhood of k_p. In fact, let us split the last expression as follows

$$\left(\frac{h_p^{-1}(k_p + o(p-2)) - h_p^{-1}(k_p)}{o(p-2)} - (h_p^{-1})'(k_p) + (h_p^{-1})'(k_p)\right) o(p-2).$$

To prove that the first part of the sum converges to 0, we only need to prove that the coefficient in front of $o(p-2)$ is uniformely bounded in p. Observe that the differentiability of h_p at 1 without control on p is not sufficient. So, let us prove that h_p is uniformely derivable in p at 1. This is true because $\lim_{(p,t)\to(0,1)} h_p'(t)$ exists, implied by Lebesgue's dominated convergence theorem jointed to assumption (AA_3) and (4.3).

For the second part, it is sufficient to prove that $(h_p^{-1})'(k_p)(p-2)$ is bounded, i.e. $\frac{h_p'(1)}{p-2}$ is away from 0. This is clear since, by assumption (AA_3), it converges to $\int_\Omega H_*(v_*)$ which is different from zero. □

We now deduce from the previous Lemma 4.1.2 the desired upper bound. We explain how just an inequality between energy and norm given by the superquadraticity assumption (AA_1) is enough.

Theorem 4.1.3. *Let $(u_p)_{p>2}$ be a family of ground state (resp. least energy nodal) solutions of Problem ($G\mathscr{P}_p$). Then, under assumptions $(GA_1)-(GA_2)$ and $(AA_1)-(AA_3)$, the family $(u_p)_{p>2}$ is bounded and if $u_{p_n} \rightharpoonup u_*$ for some sequence $p_n \to 2$, then we have*

$$\lim_{n\to\infty}\left(\frac{\mathscr{E}_{p_n}(u_{p_n})}{p_n - 2}\right) = \mathscr{E}_*(u_*) \leqslant \mathscr{E}_*(v_*),$$

for any critical point v_* of the functional $\mathscr{E}_*$ such that $\int_\Omega H_*(v_*) \neq 0$.

Proof. Let us consider the family $(\hat{v}_p)_{p>2}$ given in Lemma 4.1.2. We have that $\|\hat{v}_p\|^2 = \lambda\|\hat{v}_p\|_2^2 + o(p-2)$. Indeed, using assumption (AA_2), we infer

$$\frac{\|\hat{v}_p\|^2 - \int_\Omega f_2(\hat{v}_p)\hat{v}_p}{p-2} = \int_\Omega \frac{f_p(\hat{v}_p) - f_2(\hat{v}_p)}{p-2}\hat{v}_p \to \int_\Omega f_*(v_*)v_* = 0,$$

when $p \to 2$.

So,

$$\frac{\mathscr{E}_p(\hat{v}_p)}{p-2} = \frac{\int_\Omega(\frac{\lambda}{2}\hat{v}_p^2 - F_p(\hat{v}_p))}{p-2} + o(1) \xrightarrow[p\to 2]{} \mathscr{E}_*(v_*),$$

where the last convergence is due to (AA_2) and to the definitions of F_p and F_*.

Inequality (4.1) given by assumption (AA_1) ensures the boundedness of $(u_p)_{p>2}$ and hence the weak convergence up to a subsequence. As observed in Lemma 4.1.1, the convergence is strong. Notice also that

$$\|u_p\|^2 = \lambda\|u_p\|_2^2 + o(p-2).$$

Indeed, using assumption (AA_2) and Lemma 4.1.1, we infer

$$\frac{\|u_p\|^2 - \int_\Omega f_2(u_p)u_p}{p-2} = \int_\Omega \frac{f_p(u_p) - f_2(u_p)}{p-2}u_p \to \int_\Omega f_*(u_*)u_* = 0,$$

when $p \to 2$. Hence, arguing as for the sequence $(\hat{v}_p)_{p>2}$, we infer that

$$\lim_{n\to\infty} \frac{\mathscr{E}_{p_n}(u_{p_n})}{p_n - 2} = \mathscr{E}_*(u_*)$$

and, by definition of $(u_{p_n})_{n\in\mathbb{N}}$, we have $\mathscr{E}_{p_n}(u_{p_n}) \leqslant \mathscr{E}_{p_n}(\hat{v}_{p_n})$. This ends the proof. □

Remark 4.1.4. Assumption (AA_2) essentially ensures that we can write

$$f_p(t) = \lambda t + f_*(t)(p-2) + g_p(t)$$

where $g_p(\cdot) = o(p-2)$. Then, if we rewrite the quotient $\frac{f_p'(t)t^2 - f_p(t)t}{p-2}$ having this in mind, we obtain that a natural sufficient condition for (AA_3) to hold is that $g_p'(\cdot) = o(p-2)$ as well. In this case, the function H_* turns out to be

$$f_*'(t)t^2 - f_*(t)t$$

and since $u_* \in \mathscr{N}_*$, the condition $\int_\Omega H_*(u_*) \neq 0$ is equivalent to $\int_\Omega f_*'(u_*)u_*^2 \neq 0$ which implies that the second derivative of the energy at u_* in its own directions is not zero. This last condition is the natural starting point of an approach based on the implicit function theorem (see page 34) to ensure the existence of a continuum of solutions of Problem $(G\mathscr{P}_p)$ emanating from u_*. Let us emphasize that the arguments in this remark are very rough: we have to suppose f_* to be derivable, while in some simple model example, it is not the case everywhere on $\mathbb{R}$.

4.1.3 Lower bound

We now consider two sets of conditions which ensure that the accumulation points u_* of $(u_p)_{p>2}$ are different from 0. For the first theorem using assumption (AA_4), we need the following property, which has its own interest, about the optimal constant of the Poincaré's inequality. In any way, it is using the idea given in Chapter 3 when we are working with the q-Laplacian (see Proposition 3.2.12). We are using the "linear" version of the Krasnoselskii genus 3.3, namely Riesz-Fisher characterization.

Proposition 4.1.5. *If $u \in H_0^1(\Omega)$, then $\lambda_2 \int_\Omega (u^+)^2 \leqslant \|u^+\|^2$ or $\lambda_2 \int_\Omega (u^-)^2 \leqslant \|u^-\|^2$.*

Proof. W.l.o.g., we can assume that $u^+ \neq 0$ and $u^- \neq 0$. By the Riesz-Fisher characterization of λ_2, we have

$$\lambda_2 = \inf_{\dim F = 2} \sup_{u \in F \setminus \{0\}} \frac{\|u\|^2}{\int_\Omega |u|^2}.$$

If we define

$$F := \left\{ v \in H_0^1(\Omega) \;\middle|\; v = \alpha u^+ + \beta u^-,\ \alpha,\ \beta \in \mathbb{R} \right\},$$

then

$$\lambda_2 \leqslant \max_{(\alpha,\beta)\neq(0,0)} \frac{\alpha^2\|u^+\|^2 + \beta^2\|u^-\|^2}{\alpha^2\|u^+\|_2^2 + \beta^2\|u^-\|_2^2} = \max\left\{ \frac{\|u^+\|^2}{\|u^+\|_2^2}, \frac{\|u^-\|^2}{\|u^-\|_2^2} \right\}$$

ensuring the thesis. $\square$

In Chapter 2, Section 2.1, we studied the $\mathscr{C}$-convergence of a sequence of least energy nodal solutions. The arguments were presented for the Lane-Emden Problem ($\mathscr{P}_p$) and l.e.n.s but work, thanks to assumption (AA_2), in the same way in our framework. This convergence is one ingredient of the following Theorem 4.1.6.

Theorem 4.1.6. *Under assumptions $(GA_1)-(GA_2)$, (AA_2) and (AA_4), any bounded family $(u_p)_{p>2}$ of ground state (resp. least energy nodal) solutions is bounded away from* 0*.*

Proof. Let us denote by $(u_{p_n})_{n\in\mathbb{N}}$, where $p_n \to 2$, a weak convergent subsequence of the family $(u_p)_{p>2}$, and let u_* be its limit. We have that u_{p_n} converges to u_* uniformly. Let us show this convergence implies that $u_* \neq 0$ and more precisely that $\|u_{p_n}\|_\infty > \eta$ where η is defined in (AA_4).

In the case $\lambda = \lambda_1$, as $u_{p_n} \in \mathscr{N}_{p_n}$, we have

$$\lambda_1 \int_\Omega u_{p_n}^2 \leqslant \|u_{p_n}\|^2 = \int_\Omega f_{p_n}(u_{p_n})u_{p_n} = \int_{\Omega\setminus\{x:u_{p_n}(x)=0\}} \frac{f_{p_n}(u_{p_n})}{u_{p_n}} u_{p_n}^2, \qquad (4.4)$$

which is impossible if $\|u_{p_n}\|_\infty \leqslant \eta$.

In the case $\lambda = \lambda_2$, by Proposition 4.1.5 and since $u_{p_n}^\pm \in \mathscr{N}_{p_n}$, one of the two following inequalities hold

$$\lambda_2 \int_\Omega (u_{p_n}^+)^2 \leqslant \|u_{p_n}^+\|^2 = \int_\Omega f_{p_n}(u_{p_n}^+)u_{p_n}^+ = \int_{\Omega\setminus\{x:u_{p_n}^+(x)=0\}} \frac{f_{p_n}(u_{p_n}^+)}{u_{p_n}^+} (u_{p_n}^+)^2, \qquad (4.5)$$

$$\lambda_2 \int_\Omega (u_{p_n}^-)^2 \leqslant \|u_{p_n}^-\|^2 = \int_\Omega f_{p_n}(u_{p_n}^-)u_{p_n}^- = \int_{\Omega\setminus\{x:u_{p_n}^-(x)=0\}} \frac{f_{p_n}(u_{p_n}^-)}{u_{p_n}^-} (u_{p_n}^-)^2, \qquad (4.6)$$

which is again impossible if $\|u_{p_n}\|_\infty \leqslant \eta$. □

The second proposition generalizes the idea given in Proposition 1.1.6.

Theorem 4.1.7. *Under assumptions $(GA_1)-(GA_2)$ and (AA_4'), any family of ground state (resp. least energy nodal) solutions $(u_p)_{p>2}$ is bounded away from* 0*.*

Proof. Let us give the proof in the case $\lambda = \lambda_2$ for l.e.n.s., the other case being easier. For each $p > 2$, we define the function $\hat{v}_p$ as follows. Let $r := \frac{\int_\Omega u_p^+ e_1}{\int_\Omega |u_p| e_1} \in [0,1]$ and take

$$v_p = (1-r)u_p^+ + r u_p^- .$$

Our choice of r implies that v_p is orthogonal to e_1 in L^2. Choose now $t_p \geqslant 0$ in such a way that $\hat{v}_p := t_p v_p \in \mathscr{N}_p$. By construction, $\hat{v}_p \in E_1^\perp \cap \mathscr{N}_p$.

Since $\|\hat{v}_p\|^2 = \int_\Omega f_p(\hat{v}_p)\hat{v}_p$ because $\hat{v}_p \in \mathscr{N}_p$, we deduce by interpolation and assumption (AA_4') that, for q defined in (AA_4'),

$$\begin{aligned}\|\hat{v}_p\|^2 \leqslant & \lambda \|\hat{v}_p\|_2^{(1-\sigma)(c_1(p)+2)} \|\hat{v}_p\|_{2^*}^{\sigma(c_1(p)+2)} \\ & + c_2(p) \|\hat{v}_p\|_2^{(1-s)q} \|\hat{v}_p\|_{2^*}^{sq}\end{aligned}$$

where $s, \sigma \in [0,1]$ are respectively equal to $\frac{q-2}{q} \cdot \frac{2^*}{2^*-2}$ and to the same quantity with q replaced by $c_1(p)+2$. Using Sobolev's inequalities, the variational characterization of λ_2, and the fact that $\hat{v}_p \in E_1^\perp$, we now infer that

$$\|\hat{v}_p\|^2 \leqslant \lambda^{1-\frac{(1-\sigma)(c_1(p)+2)}{2}} S^{\sigma(c_1(p)+2)} \|\hat{v}_p\|^{c_1(p)+2} + c_2(p) \lambda^{\frac{(s-1)q}{2}} S^{sq} \|\hat{v}_p\|^q .$$

We then have

$$1 \leqslant \alpha\beta \|\hat{v}_p\|^{c_1(p)} + c_2(p) ab \|\hat{v}_p\|^{q-2}$$

where

$$\alpha(p) := \lambda^{1-\frac{(1-\sigma)(c_1(p)+2)}{2}}, \quad \beta(p) := S^{\sigma(c_1(p)+2)}, \quad a := \lambda^{\frac{(s-1)q}{2}}, \quad b := S^{sq}.$$

It implies that $\hat{v}_p$ stays away from 0. If not, by contradiction, $\|\hat{v}_p\| \to 0$. We deduce that, for p sufficiently close to 2,

$$\left(\frac{1-c_2(p)}{\alpha\beta}\right)^{\frac{1}{c_1(p)}} \leq \left(\frac{1-c_2(p)ab\|\hat{v}_p\|^{q-2}}{\alpha\beta}\right)^{\frac{1}{c_1(p)}} \leqslant \|\hat{v}_p\|,$$

which will lead to a contradiction if we prove that the left-hand side of the inequality does not converge to 0. This is the case since assumption (AA_4') ensures that the numerator does not converge to 0 and since the denominator stays bounded.

We then conclude as in Proposition 1.1.6 that $\|\hat{v}_p\| \leq \|u_p\|$ to finish the proof. □

4.2 Symmetries

4.2.1 Uniqueness around an eigenfunction

In this section, we prove that, under assumptions (SA_1) and (SA_2), for p close to 2, any bounded family $(u_p)_{p>2}$ of solutions staying away from zero respects the symmetries of the family projected in E. It follows from a uniqueness result (up to projection) which is our next concern. We emphasize that the result holds at any eigenvalue λ_i of $-\Delta$. It generalizes Lemma 1.4.1.

Lemma 4.2.1. *Let $N \geqslant 2$. There exists $\eta > 0$ such that, if $\|a(x) - \lambda_i\|_{N/2} < \eta$ and u solves*

$$\begin{cases} -\Delta u = a(x)u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{R}_p$}$$

then, either $u = 0$ or $P_{E_i} u \neq 0$.

Proof. Let $N \geqslant 3$ and assume that there exists a non-zero solution u such that $P_{E_i} u = 0$. Let $w := P_{E_1 \oplus \ldots \oplus E_{i-1}} u$ and $z := P_{(E_1 \oplus \ldots \oplus E_i)^\perp} u$. There exists $C > 0$ such that

$$\|w\|^2 \geqslant \frac{\lambda_i}{\lambda_{i-1}} \|w\|^2 - C\|a(x) - \lambda_i\|_{L^{\frac{N}{2}}} \|w\| \|u\|,$$

and

$$\|z\|^2 \leqslant \frac{\lambda_i}{\lambda_{i+1}} \|z\|^2 + C\|a(x) - \lambda_i\|_{L^{\frac{N}{2}}} \|z\| \|u\|.$$

So,

$$\|w\| \leqslant \frac{\lambda_{i-1} C}{\lambda_i - \lambda_{i-1}} \|a(x) - \lambda_i\|_{\frac{N}{2}} \|u\| \text{ and } \|z\| \leqslant \frac{\lambda_{i+1} C}{\lambda_{i+1} - \lambda_i} \|a(x) - \lambda_i\|_{\frac{N}{2}} \|u\|.$$

The conclusion now easily follows.

For $N = 2$, the same statement can be formulated with the $L^{N/2}$-norm replaced by L^q-norm for some $1 < q < +\infty$. □

Proposition 4.2.2. *Assume that $(GA_1) - (GA_2)$ and $(SA_1) - (SA_2)$ hold and $(u_{1,p})_{p>2}, (u_{2,p})_{p>2}$ are families of solutions of Problem $(G\mathscr{P}_p)$. For every $M > 0$, there exists $p_M > 2$ such that, if $\|u_{i,p}\| \leqslant M$ and $\|P_E(u_{i,p})\| \geqslant 1/M$ for $2 < p < p_M$, then either $u_{1,p} = u_{2,p}$ or $P_E u_{1,p} \neq P_E u_{2,p}$ for every $2 < p < p_M$, where E is the eigenspace associated with the eigenvalue λ of assumption (SA_1).*

Proof. Idea is similar as for Proposition 1.4.2. We just obtain

$$a_n(x) = \begin{cases} \dfrac{f_{p_n}(u_{1,p_n}(x)) - f_{p_n}(u_{2,p_n}(x))}{u_{1,p_n}(x) - u_{2,p_n}(x)}, & \text{if } u_{1,p_n}(x) \neq u_{2,p_n}(x), \\ \lambda, & \text{otherwise.} \end{cases}$$

Assumption (SA_2) ensures that Lebesgue's dominated convergence theorem applies and since u_{i,p_n} converge to a non-zero function almost everywhere, we deduce that $a_n \to \lambda$ in $L^{N/2}(\Omega)$. The proof now follows from Proposition 4.2.1. $\square$

From what precedes, we can deduce an abstract symmetry result.

Theorem 4.2.3. *Let $(G_\alpha)_{\alpha \in E}$ be groups acting on $H_0^1(\Omega)$ in such a way that, for every $g \in G_\alpha$ and for every $u \in H_0^1(\Omega)$,*

$$(i) g(E) = E, \quad (ii) g(E^\perp) = E^\perp, \quad (iii) g\alpha = \alpha, \quad (iv) \mathscr{E}_p(gu) = \mathscr{E}_p(u).$$

Then, under assumptions $(GA_1)-(GA_2)$ and $(SA_1)-(SA_2)$, for all $M > 0$, if p is close enough to 2, any solution u_p of Problem $(G\mathscr{P}_p)$ satisfying $\|u_p\| \leqslant M$ and $\|P_E(u_p)\| \geqslant 1/M$ belongs to the invariant set of G_{α_p} where $\alpha_p := P_E u_p$.

Proof. The proof is similar as Theorem 1.4.5. $\square$

4.2.2 Symmetry breaking of least energy nodal solutions on rectangles

In Chapter 1, we proved that the symmetries of least energy nodal solutions for the Lane–Emden model $(\mathscr{P}_p)$ cannot be extended to large p. We exhibited least energy nodal solutions on a rectangle that are neither symmetric nor antisymmetric with respect to the medians. Arguing in the same way requires extending the previous results to a more general second order elliptic operator. Namely, one can first prove that the previous results concerning Problem $(G\mathscr{P}_p)$ extend to the similar boundary value problem with $-\Delta$ replaced by $-\operatorname{div}(A_p \nabla)$ where $A_p \in \mathscr{C}(\Omega, S^{N\times N})$, $S^{N\times N}$ is the set of symmetric $N \times N$ matrices, $A_0 = \mathrm{id}$ and $p \mapsto A_p$ is uniformly differentiable at $p = 0$. Nothing changes in the previous proofs to obtain the same conclusions, except for the non-convergence to 0.

Indeed, the modifications in the proof of Theorem 4.1.7 are straightforward as soon as we assume the extra condition

$$\exists k \in \mathbb{R} : \forall p > 2 : \frac{\log(1 - c_2(p) - O(p-2))}{c_1(p)} \geqslant k. \tag{4.7}$$

As we are working on rectangle, we obtain the $\mathscr{C}$-convergence.

When dealing with (AA_4) in Theorem 4.1.6, we require that $A_p - \mathrm{id}$ is semi positive definite. Then, we control the sign of the additional term in the right-hand side of (4.4) (or (4.5) and (4.6))

$$\|u_{p_n}\|^2 = \int_{\Omega \setminus \{x : u_{p_n}(x) = 0\}} \frac{f_{p_n}(u_{p_n})}{u_{p_n}} u_{p_n}^2 - \int_\Omega \nabla u_{p_n} \cdot (A_{p_n} - \mathrm{id}) \nabla u_{p_n}.$$

We next quickly sketch the argument giving the symmetry breaking on rectangles and check that the last extra assumption (4.7) is fulfilled in this situation. Consider sequences $(p_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $0 < \varepsilon_n = o(p_n - 2)$. Let R_{ε_n} be rectangles with sides of length 2 and $(1-\varepsilon_n)2$. We then consider the sequence of problems

$$\begin{cases} -\Delta u = f_{p_n}(u), & \text{in } R_{\varepsilon_n}, \\ u = 0, & \text{on } \partial R_{\varepsilon_n}. \end{cases} \tag{$\mathscr{P}_{\varepsilon_n}$}$$

The change of variable $\tilde{u}(x,y) = u(x,(1-\varepsilon)y)$ leads to the equivalent sequence of problems

$$\begin{cases} -\partial_x^2 \tilde{u} - (1-\varepsilon_n)^{-2} \partial_y^2 \tilde{u} = f_{p_n}(\tilde{u}), & \text{in } Q, \\ \tilde{u} = 0, & \text{on } \partial Q, \end{cases}$$

where Q denotes the square $(-1,1)^2$. In both cases, $\lambda = \lambda_2(Q)$. Observe that the differential operator can be written in divergence form with

$$A_{p_n} = \begin{pmatrix} 1 & 0 \\ 0 & (1-\varepsilon_n)^{-2} \end{pmatrix}$$

so that $A_{p_n} - \mathrm{id}$ is semi positive definite.

So, we obtain a family of problems exhibiting a symmetry breaking.

Theorem 4.2.4. *Assume $\lambda = \lambda_2$ and f_p fulfills the assumptions $(GA_1) - (GA_2)$, $(AA_1) - (AA_4)$, $(SA_1) - (SA_2)$ and the extra condition (4.7). If Ω is a square* *and if the minimizers u_* of $\mathscr{E}_*$ on $\mathscr{N}_*$ verifies $\int_\Omega H_*(u_*) \neq 0$ and are neither symmetric nor antisymmetric with respect to the medians, then there exists a rectangle R and $p > 2$ such that any least energy nodal solutions of the problem*

$$\begin{cases} -\Delta u = f_p(u), & \text{in } R, \\ \quad u = 0, & \text{on } \partial R, \end{cases}$$

is neither symmetric nor antisymmetric with respect to the medians of R.

Assumption (GA_1) can be replaced by (GA_1') in dimension 2 while (AA_4) can be substituted by (AA_4').

4.3 Examples: explicit nonlinearities

In this section, we study three examples of problem different from the Lane–Emden one $(\mathscr{P}_p)$. All cases will be illustrated on the square $\Omega = (-1,1)^2$ in $\mathbb{R}^2$. Let us recall that

$$\lambda_1 = \frac{\pi^2}{2} \quad \text{and} \quad \lambda_2 = \frac{5\pi^2}{4}.$$

We will also study the presence of a symmetry breaking as described in Theorem 4.2.4. To this aim, we need to compute the function $\mathscr{E}_*$ on $\mathscr{N}_*$. Let us recall that

$$E_2 = \operatorname{span}\{v_1, v_2\},$$

where

$$v_1(x,y) = \cos\left(\frac{\pi}{2}x\right)\sin(\pi y) \quad \text{and} \quad v_2(x,y) = \sin(\pi x)\cos\left(\frac{\pi}{2}y\right).$$

4.3.1 Superlinear perturbation of Lane–Emden Problem

The first example is a very natural generalization of Problem $(\mathscr{P}_p)$. Let us consider the function

$$f_p(t) = \lambda t|t|^{p-2} + (p-2)t|t|^{q-2}$$

for some $q \in (2, 2^*)$. We then check that

- $(GA_1) - (GA_2)$ is clearly satisfied;

- (AA_1) is clearly verified;
- $(AA_2)-(AA_3)$, $(SA_1)-(SA_2)$ are verified by $f_*(t) = \lambda t \log|t| + t|t|^{q-2}$, $H_*(t) = \lambda t^2 + (q-2)|t|^q$ and $h(t) = (2\lambda + q - 1)(|t|^s + 1)$, with $s = q-1$ or $q-2$ accordingly. Let us just remark that every non-zero critical point u_* of $\mathscr{E}_*$ verifies $\int_\Omega H_*(u_*) \neq 0$;
- (AA_4) is verified for $c_1(p) = c_2(p) = p-2$.

In order to illustrate this example numerically, let us consider the following problem

$$\begin{cases} -\Delta u = \lambda u^3 + 2u^5, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.8}$$

for $\lambda = \lambda_1$ or λ_2. It is worth pointing out that instead of considering Problem (4.8), we could have dealt with the model

$$\begin{cases} -\Delta u = u^3 + cu^5, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases}$$

as there is an obvious bijection between the solutions of this last problem (with a precise choice of the constant c) and the solutions of Problem (4.8). As already mentioned, from a theoretical point of view, dealing with $f_p(t)$ instead of $t|t|^{p-2} + (p-2)t|t|^{q-2}$ allows to avoid recurrent rescalings that make the correct limit equation to appear. From the numerical point of view, looking at the rescaled equation where the eigenvalue appears is the best choice to avoid very small or very large solutions that could lead to numerical inaccuracy and would not facilitate the visualization.

Figure 4.1 and Table 4.1 show the results of the numerical experiments using MPA and MMPA: one-signed (resp. nodal) numerical solutions have the expected symmetries.

Concerning least energy nodal solutions, the nodal line seems to be a diagonal. Moreover, minimizers of $\mathscr{E}_*$ on $\mathscr{N}_*$ seems to be symmetric with respect to a diagonal. Indeed, any second eigenfunctions can be written as a multiple of $v_\alpha = \cos(\alpha)v_1 + \sin(\alpha)v_2$, for $\alpha \in [0, 2\pi]$, where v_1 and v_2 have been defined previously. Figure 4.2 approaches the graph of the entropy function

$$S : [0, 2\pi] \to \mathbb{R} : \alpha \mapsto \mathscr{E}_*(t_\alpha v_\alpha),$$

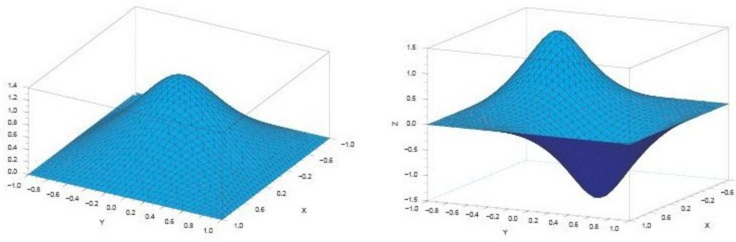


Figure 4.1: Numerical solutions in the case of a superlinear perturbation of the Lane–Emden Problem.

	Initial function	$\min u$	$\max u$	$\mathcal{E}_4(u^+)$	$\mathcal{E}_4(u^-)$
λ_1	$\sin(\pi(x+1)/2)\sin(\pi(y+1)/2)$	0.0	1.29	1.47	0.0
λ_2	$\sin(\pi(x+1))\sin(2\pi(y+1))$	−1.4	1.4	1.7	1.7

Table 4.1: Characteristics of the solutions for a superlinear perturbation.

where $t_\alpha > 0$ is such that $v_\alpha \in \mathcal{N}_*$, i.e. t_α is a solution of $\frac{1}{t}\int_\Omega f_*(tv_\alpha)v_\alpha = 0$. As the minima seems to be attained for $\alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$, the corresponding minimizers should be symmetric functions with respect to a diagonal. As a consequence of the previous computation, a symmetry breaking on some rectangles described in Theorem 4.2.4 occurs.

4.3.2 Exponential growth

The second example is a nonlinearity with an exponential growth in dimension 2:

$$f_p(t) = \lambda t(e^{t^2} - 1)^{p-2}.$$

We observe that

- $(GA_1') - (GA_2)$ are satisfied for some $\gamma > 0$;

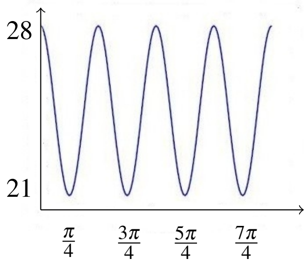


Figure 4.2: Entropy S in the superlinear perturbation case.

- (AA_1) is verified. Indeed, we have

$$\begin{aligned}(f_p(t)t)' - (pF_p(t))' &= t(f_p(t))' - (p-1)f_p(t)\\ &= \lambda t(e^{t^2}-1)^{p-2}\left(2(p-2)\frac{t^2e^{t^2}}{e^{t^2}-1} + 2 - p\right)\end{aligned}$$

  which proves the claim because $\frac{2t^2e^{t^2}}{e^{t^2}-1} \geqslant 1$ for all $t \in \mathbb{R}$;

- $(AA_2)-(AA_3), (SA_1)-(SA_2)$ are verified by $f_*(t) = \lambda t \log(e^{t^2}-1)$, $H_*(t) = 2\lambda \frac{t^4e^{t^2}}{e^{t^2}-1}$ (extended by 0 at 0) and $h(t) = C(\delta)e^{(\delta+1)t^2}$, for some $\delta > 0$ and $C(\delta) > 0$;

- (AA_4') is verified by taking for example $\eta = \sqrt{\log(a)}$ with $1 < a < 2$.

We numerically illustrate this example with the model problem

$$\begin{cases} -\Delta u = \lambda u(e^{u^2}-1)^{0.5}, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Figure 4.3 and Table 4.2 show the results of the numerical experiments. The same comments as in the previous example can be made. In particular, in the case of least energy nodal solutions, minimizers of $\mathscr{E}_*$ on $\mathscr{N}_*$ seems to be symmetric with respect to a diagonal as shown in Figure 4.4 and the symmetry breaking described in Theorem 4.2.4 still occurs in this example. We used a secant method to compute the scaling factor for projecting on $\mathscr{N}_*$. Let us

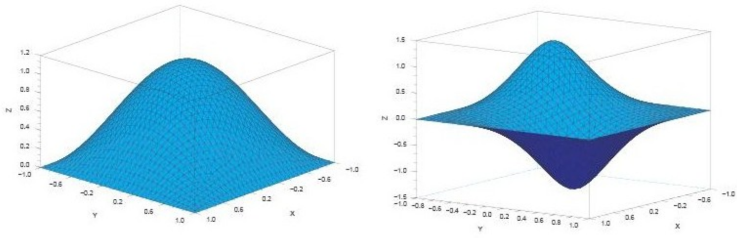


Figure 4.3: Numerical solutions in the case of an exponential growth.

mention that, to avoid a “blow-up” of the roundoff error, we used the function `expm1(x)` to compute $e^x - 1$. The expression G denotes ground state solution and L denotes least energy nodal solution on tables.

	Initial function	$\min u$	$\max u$	$\mathscr{E}_{2.5}(u^+)$	$\mathscr{E}_{2.5}(u^-)$
G	$\sin(\pi(x+1)/2)\sin((y+1)/2)$	0.0	1.1	3.0	0.0
L	$\sin(\pi(x+1))\sin(2\pi(y+1))$	-1.2	1.2	2.7	2.7

Table 4.2: Characteristics of the solutions for an exponential growth.

4.3.3 Sum of powers

In this example, for technical reasons, we consider a family $(p_\varepsilon)_{\varepsilon>0}$ converging to 2 when $\varepsilon \to 0$ and rename $f_{p_\varepsilon}, \mathscr{E}_{p_\varepsilon}$ by $f_\varepsilon, \mathscr{E}_\varepsilon$. We consider the functions

$$f_\varepsilon(t) = \lambda t \left(\sum_{i=1}^{k} \alpha_i |t|^{\beta_i(\varepsilon)} \right)$$

with $\alpha_i, \beta_i(\varepsilon) > 0$ such that $\min \beta_i(\varepsilon) = \beta_1(\varepsilon)$, $\max \beta_i(\varepsilon) = \beta_k(\varepsilon)$, $\sum_i \alpha_i = 1$, $\lim_{\varepsilon\to 0} \beta_i(\varepsilon) = 0$ and $\lim_{\varepsilon\to 0} \frac{\beta_k(\varepsilon)}{\beta_1(\varepsilon)} = 1$. We then check that

- $(GA_1) - (GA_2)$ are clearly satisfied with $q = \beta_k(\varepsilon)$;
- (AA_1) holds with $p_\varepsilon = \beta_1(\varepsilon) + 2$;

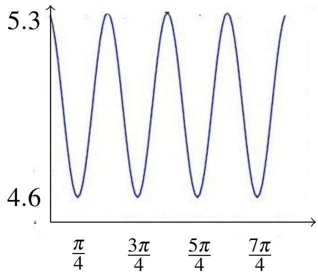


Figure 4.4: Entropy S in the case of an exponential growth.

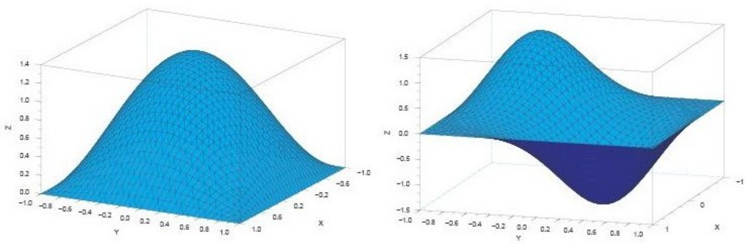


Figure 4.5: Numerical solutions in the case of a sum of powers.

- $(AA_2)-(A_3),(SA_1)-(SA_2)$ are verified by $f_*(t)=\lambda t\log|t|, H_*(t)=\lambda t^2$ and $p=1+\delta$ for some $\delta>0$;
- (AA_4') is verified for some $\eta<1$.

We illustrate this example numerically by considering

$$\begin{cases} -\Delta u=\lambda u\big(\frac{1}{4}|u|^{0.25}+\frac{1}{4}|u|^{0.5}+\frac{1}{2}|u|\big), & \text{in } \Omega,\\ \quad u=0, & \text{on } \partial\Omega. \end{cases}$$

Figure 4.5 and Table 4.3 show the results of the numerical experiments. We can do the same comments as in the previous cases. Let us denote $r=\sqrt{x^2+y^2}$.

Since the limit functional for this example is the same as the limit functional for the Lane–Emden Problem $(\mathscr{P}_p)$, the symmetry breaking phenomenum occurs.

	Initial Function	$\min u$	$\max u$	$\mathscr{E}_1(u^+)$	$\mathscr{E}_1(u^-)$
λ_1	$\cos(\pi r/2)$	0.0	1.3	4.7	0.0
λ_2	$-\cos\sqrt{\pi/2}r\cos\sqrt{\pi}r$	-1.4	1.4	4.6	4.6

Table 4.3: Characteristics of the solutions for the sum of powers.

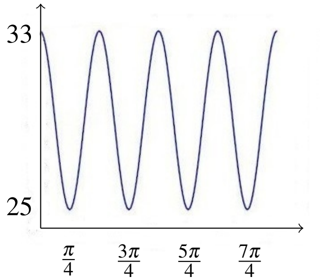


Figure 4.6: Entropy S in the case of a sum of powers.

4.4 Super-lower homogeneity assumption

In this last section, we provide an assumption of "*Super-lower homogeneity*" that leads to a naturel boundedness of any family of ground state (resp. least energy nodal) solutions for Problem ($G\mathscr{P}_p$). This result has to be compared with Theorem 4.1.3: its assumptions cover other nonlinearities than assumption (AA_3). The idea is inspired by Proposition 1.1.2.

Proposition 4.4.1. *Let us assume assumptions* $(AA_2)(a),(b)$ *and*

$$\exists \rho \geqslant 0,\ \forall s \in \mathbb{R},\ \forall t \geqslant \rho :\ f_p(ts) \geqslant \sigma(s)t^{p-1} f_p(s),$$

where $\sigma(s)$ *is the sign*[1] *of* s*. Then, there exists a bounded family* $(v_p)_{p>2}$ *in* $H_0^1(\Omega)$ *such that, for any* p*,* $v_p \in \mathscr{N}_p$ *(resp.* $\mathscr{M}_p$*).*

Proof. We only consider the case of $\mathscr{M}_p$, the other one being similar. It is sufficient to prove that if $p_n \to 0$ then there exists a bounded sequence $v_{p_n} \in \mathscr{M}_{p_n}$.

[1] The sign of s is defined by $\sigma(s) = 1$ if $s > 0$, $\sigma(s) = 0$ if $s = 0$ and $\sigma(s) = -1$ if $s < 0$.

Take $e_2 \in E_2$ and define v_p as $t_{p_n}^+ e_2^+ + t_{p_n}^- e_2^-$ where $t_{p_n}^\pm$ are such that $t_{p_n}^\pm e_2^\pm \in \mathscr{N}_p$. Without loss of generality, we can suppose that $t_{p_n}^\pm$ are away from 0 (otherwise we are done). So, up to a scaling of e_2, we can assume that $t_{p_n}^\pm \geqslant \rho$ for all n. The assumption ensures that

$$(t_{p_n}^\pm)^2 \|e_2^\pm\|^2 = \int_\Omega f_{p_n}(t_{p_n}^\pm e_2^\pm) t_{p_n}^\pm e_2^\pm \geqslant \int_\Omega (t_{p_n}^\pm)^{p_n} f_{p_n}(e_2^\pm) e_2^\pm$$

which implies that

$$(t_{p_n}^\pm)^{p_n-2} \leqslant \frac{\|e_2^\pm\|^2}{\int_\Omega f_{p_n}(e_2^\pm) e_2^\pm}.$$

As $\|e_2^\pm\|^2 = \lambda_2 \|e_2^\pm\|_2^2$ and $f_{p_n}(e_2^\pm) \to \lambda e_2^\pm$, we just need to study the convergence of the last term in the expression

$$-\frac{\log(\|e_2^\pm\|^2) - \log(\int_\Omega f_{p_n}(e_2^\pm) e_2^\pm)}{\|e_2^\pm\|^2 - \int_\Omega f_{p_n}(e_2^\pm) e_2^\pm} \cdot \int_\Omega \frac{f_{p_n}(e_2^\pm) e_2^\pm - \lambda_2 (e_2^\pm)^2}{p_n - 2}$$

since the first one converges to $-\|e_2^\pm\|^{-2}$, as in Section 1.1. The assumptions $(AA_2)(a), (b)$ ensuring that the last term converges to $\int_\Omega f_*(e_2^\pm) e_2^\pm$, we deduce the boundedness of $(t_{p_n}^\pm)_{n\in\mathbb{N}}$ and hence of v_{p_n}. $\square$

Observe now that Proposition 4.4.1 leads in a straightforward way to an alternative assumption ensuring the boundedness of ground state (resp. least energy nodal) solutions of Problem $(G\mathscr{P}_p)$. With this alternative, we can treat the case of the sum of powers without the assumption on $\beta_k(\varepsilon)\beta_1(\varepsilon)^{-1}$. If $t \geqslant 1$, we have

$$\begin{aligned}\lambda \sum_{i=1}^n \left(\alpha_i (ts)^{\beta_i(\varepsilon)} ts\right) &\geqslant \lambda \min_{i\in\{1,\dots,n\}} (t^{\beta_i(\varepsilon)+1}) \left(\sum_{i=1}^n \alpha_i s^{\beta_i(\varepsilon)+1}\right) \\ &= \lambda t^{(\beta_1(\varepsilon)+1)} \left(\sum_{i=1}^n \alpha_i s^{\beta_i(\varepsilon)} s\right)\end{aligned}$$

and we can conclude recalling that $p_\varepsilon = \beta_1(\varepsilon) + 2$. In fact, we have a better control on the Nehari manifold. Namely,

$$\left(\frac{1}{2} - \frac{1}{\beta_1(\varepsilon)+2}\right) \|u\|^2 \leqslant \mathscr{E}_\varepsilon(u) \leqslant \left(\frac{1}{2} - \frac{1}{\beta_k(\varepsilon)+2}\right) \|u\|^2$$

for every $u \in \mathscr{N}_\varepsilon$.

Chapter 5

Lane–Emden problem with NBC

In this chapter, we work with the Lane–Emden Problem with Neumann boundary conditions ($N\mathscr{P}_p$)

$$\begin{cases} -\Delta u + u = \lambda |u|^{p-2}u, & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$N\mathscr{P}_p$}$$

where $\lambda > 0$[1]. *Let us just remark that we work with* $V \equiv 1$ *to avoid zero being the first eigenvalue of the left-hand side of the equation. So, the operator* $-\Delta + V$ *stays positive definite. In this way, the variational formulation has a mountain pass geometry. The weak solutions of Problem* ($N\mathscr{P}_p$) *are the critical points of the energy functional* $\mathscr{E}_p$ *defined on the classical Sobolev space*[2] $H^1(\Omega)$ *and given by*

$$\mathscr{E}_p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + u^2 - \frac{\lambda}{p}\int_\Omega |u|^p.$$

As for DBC, energy functional $\mathscr{E}_p$ *is of class* $\mathscr{C}^2(\Omega)$ *and satisfies the Palais-Smale condition. Ground state (resp. least energy nodal) solutions exist and*

[1] If Ω is not of class $\mathscr{C}^1$ then the boundary conditions must be perform with the trace operator.

[2] Functions in $L^2(\Omega)$ with weak derivative belonging to $L^2(\Omega)$, with associated norm $\|u\|^2 = \int_\Omega |\nabla u|^2 + u^2$.

are minima of the energy functional $\mathscr{E}_p$ *on the Nehari manifold* $\mathscr{N}_p$ *(resp. nodal Nehari set* $\mathscr{M}_p$*) defined by*

$$\mathscr{N}_p := \{u \in H^1(\Omega) \setminus \{0\} : \langle \mathrm{d}\mathscr{E}_p(u), u\rangle = 0\}, \quad \mathscr{M}_p := \{u \in H^1(\Omega) : u^{\pm} \in \mathscr{N}_p\},$$

where

$$\langle \mathrm{d}\mathscr{E}_p(u), v\rangle = \int_\Omega \nabla u \nabla v + uv - \lambda \int_\Omega |u|^{p-2}uv.$$

We have that a ground state solution (resp. l.e.n.s) must not change sign (resp. possess two nodal domains). These facts can be proven using the same arguments as for the DBC (see Introduction of Chapter 1*). For any* $u \in H^1(\Omega)$*,* $u \in \mathscr{N}_p$ *if and only if*

$$\int_\Omega |\nabla u|^2 + u^2 = \lambda_2 \int_\Omega |u|^p, \tag{5.1}$$

which implies that $\left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 = \mathscr{E}_p(u)$*. We also have that, for any* $u \in H^1(\Omega) \setminus \{0\}$*, there exists one and only one positive multiplicative factor* $t^* > 0$ *such that* $t^*u \in \mathscr{N}_p$*. As zero is a strict local minimum of* $\mathscr{E}_p$*, it implies that* $\mathscr{E}_p(t^*u) = \max_{t>0}(\mathscr{E}_p(tu))$*. If* $u \in H^1(\Omega)$*,* $u^+ \neq 0$ *and* $u^- \neq 0$*, then* $u \in \mathscr{M}_p$ *if and only if*

$$\int_\Omega |\nabla u^+|^2 + (u^+)^2 = \lambda \int_\Omega |u^+|^p \text{ and } \int_\Omega |\nabla u^-|^2 + (u^-)^2 = \lambda \int_\Omega |u^-|^p. \tag{5.2}$$

At $u \in H^1(\Omega) \setminus \{0\}$*, we can project it on* $\mathscr{M}_p$ *by maximizing the energy functional on the quarter of plane defined by* u^+ *and* u^-*.*

To start, in Section 5.1, we show that techniques and results developed in Chapter 1 *are also working for Neumann boundary conditions. Let us denote* λ_i *(resp.* E_i*) the eigenvalues without multiplicity (resp. eigenspaces) of* $-\Delta + \mathrm{id}$ *with NBC. By homogeneity, without loss of generality, we can assume that* $\lambda = \lambda_1$ *(resp.* λ_2*) when we are working with ground state solutions (resp. l.e.n.s.). In particular, we obtain that ground state solutions must respect the symmetries of* Ω *for* p *close to* 2*. It is an alternative to the Gidas, Ni and Nirenberg method which is only applicable for DBC. Moreover, ground state solutions must be the non-zero constant solutions for* p *close to* 2*. In fact, we show the uniqueness of the positive solution for* p *close to* 2*. Concerning least energy nodal solutions, we obtain similar symmetries as for DBC when* Ω *is a rectangle, a radial domain or a square. We also obtain some symmetry breaking on some rectangles.*

Then, in Section 5.2, we focus on larger p and ground state solutions. We study for which p we obtain the first symmetry breaking whose the existence is implied by the results in [62] (see Introduction page 25), i.e. we are looking for the first p where ground state solutions are not a constant function anymore. For this, in Section 5.2.1, we need to work with the notion of Morse index of $\mathscr{E}_p$ at the non-zero constant solutions. For a critical point u of an application F, the Morse index is defined as

$$\operatorname{ind}_M(u,F) := \sup\{\dim E : \dim E < +\infty, \langle \mathrm{d}^2F(u), v, v\rangle < 0 \textit{ if } v \in E \setminus \{0\}\}.$$

It corresponds to the number of independent descent directions of the second derivative. As a ground state solution u minimizes the energy functional on $\mathscr{N}_p$, a manifold of co-dimension 1 *composed of local maxima, we have that* $\operatorname{ind}_M(u,\mathscr{E}_p) = 1$. *The descent direction is given by u itself. As consequence, we prove that non-zero constant solutions cannot be ground state solutions any-more for $p > 1+\lambda_2$. To know whether $1+\lambda_2$ is optimal, we analyze bifurcations starting from the non-zero constant solutions in Section 5.2.2. In particular, we obtain bifurcations when p is equal to $1+\lambda_i$. For this part, we use the following traditional Krasnoselskii-Boehme-Marino theorem. To understand well what we mean by bifurcation, let us consider a continuous function*

$$F : \mathbb{R} \times H \to H : (t,u) \mapsto F(t,u),$$

where H is a Banach space. Assume that there exists $v \in H$ such that, for any $t \in \mathbb{R}$, $F(t,v) = 0$. In this context, let us define notions of bifurcation:

- *$(\lambda, v) \in \mathbb{R} \times H$ is a bifurcation point for the equation $F(t,u) = 0$ if and only if there exists a sequence $(\lambda_n, v_n) \to (\lambda, v)$, $v_n \neq v$, such that, for any $n \in \mathbb{N}$, $F(\lambda_n, v_n) = 0$.*

  In this case, sequence $(v_n)_{n\in\mathbb{N}}$ is called bifurcation sequence.

- *$(\lambda, v) \in \mathbb{R} \times H$ is a continuous bifurcation point for $F(t,u) = 0$ if and only if there exists a non-trivial continuum[3] of solutions for equation $F(t,u) =$ 0 containing (λ, v).*

[3]Connected domain.

In this last case, the branch is called bifurcation branch. If it is "maximal"[4]*, the branch (resp. point) is called global bifurcation branch (resp. global bifurcation point).*

Krasnoselskii-Boehme-Marino Theorem.
Let $F : I \times H \to H : (t,u) \mapsto F(t,u)$ *be a continuous function, where* $I \subseteq \mathbb{R}$ *is an interval and* H *is a Banach space, such that* $F(\lambda,0) = 0$ *for any* $\lambda \in I$.

- *If* F *is of class* $\mathscr{C}^1$ *in a neighbourhood of* $(\lambda,0)$ *and* $(\lambda,0)$ *is a bifurcation point of* F *then* $F'_u(\lambda,0)$ *is not invertible.*
- *Let assume that for each* $(\lambda,u) \in I \times H$,

  $$F(\lambda,u) = L(\lambda,u) - N(\lambda,u), \quad L(\lambda,\cdot) = \lambda \,\mathrm{id} - T \text{ and } N(\lambda,u) = o(\|u\|), \tag{5.3}$$

  with T *linear,* T *and* N *completely continuous, and the last equality being uniform on each compact set of* λ.

  If λ_* *is an eigenvalue of* T *with odd multiplicity, then* $(\lambda_*,0)$ *is a global bifurcation point for* $F(t,u) = 0$.
- *Let assume that* H *is a Hilbert space and that for each* $(\lambda,u) \in I \times \mathbb{R}$, $F(\lambda,u) = \nabla_u h(\lambda,u)$ *where*

  $$\begin{gathered} h(\lambda,u) = \langle L(\lambda,u),u\rangle/2 - g(\lambda,u), \quad L(\lambda,\cdot) = \lambda \,\mathrm{id} - T, \\ \text{and } \nabla g(\lambda,u) = o(\|u\|), \end{gathered} \tag{5.4}$$

  with T *linear and symmetric,* $g(\lambda,\cdot) \in \mathscr{C}^2$ *for all* λ*, and the last equality being uniform on each compact set of* λ.

  If λ_* *is an eigenvalue of* T *with finite multiplicity and* $h(\lambda,\cdot)$ *verifies the Palais-Smale condition for each* λ*, then* $(\lambda_*,0)$ *is a bifurcation point for* $F(t,u) = 0$.

*The first point is the contraposition of the implicit function theorem (**see page 34**). The second point, proved by Krasnoselskii, is based on the degree theory and the third one follows from the Morse theory (see e.g. [21, 50]).*

[4] The branch cannot be extended. Let us remark that each bifurcation branch can be extended to a "maximal" branch.

Once the bifurcations are obtained, we focus on balls. We study in Section 5.2.3 whether bifurcations are radial or not.

In this, an important tool is the Rabinowitz's principle (see e.g. [60]).

Rabinowitz principle.
Bifurcations given by the second point of the Krasnoselskii-Boehme-Marino theorem are unbounded or linked together by pair.

Moreover, by adapting result of A. Aftalion and F. Pacella (see e.g. [3]), we obtain that any radial ground state solution must be a constant solution.

Finally, in Section 5.2.4, we illustrate this numerically and we conjecture that there is a non-radial symmetry breaking of ground state solutions when p equals $1+\lambda_2$, on a ball and in dimension $N = 2$.

This work [13] is a collaboration with D. Bonheure and V. Bouchez.

5.1 Asymptotic symmetries when $p \to 2$

Let us fix $(u_p)_{p>2}$ a family of ground state solutions (resp. l.e.n.s.) and $\lambda = \lambda_1$ (resp. λ_2).

5.1.1 Asymptotic behavior

An upper bound for $(u_p)_{p>2}$ can be obtained as for Proposition 1.1.3. We just need to work with the space $H^1(\Omega)$ in the Fredholm alternative. A lower bound can also be deduced as for Proposition 1.1.5. We can adapt Poincaré's inequalities by working with traditional Poincaré-Wirtinger's inequalities.

So, we obtain the same conclusion about asymptotic behavior as in Theorem 1.1.8.

Theorem 5.1.1. *Let $(u_p)_{p>2}$ be a family of ground state (resp. least energy nodal) solutions of Problem $(N\mathscr{P}_p)$. If $u_{p_n} \rightharpoonup u_*$ in $H^1(\Omega)$ for a sequence $p_n \to 2$, then $u_{p_n} \to u_*$ in $H^1(\Omega) \setminus \{0\}$, u_* is such that*

$$\begin{cases} -\Delta u_* + u_* = \lambda u_*, & \text{in } \Omega, \\ \partial_\nu u_* = 0, & \text{on } \partial\Omega, \end{cases}$$

with $\lambda = \lambda_1$ (resp. λ_2), and

$$\mathscr{E}_*(u_*) = \inf\{\mathscr{E}_*(u) : u \in E \setminus \{0\} \text{ and } \langle \mathrm{d}\mathscr{E}_*(u), u\rangle = 0\},$$

with $E = E_1$ (resp. E_2), the first (resp. second) eigenspace of $-\Delta + \mathrm{id}$.

5.1.2 Symmetries

For this part, we are directly working in the same way as the "general technique" described in Section 1.4. To obtain the uniqueness result 1.4.1, we just need as previously to adapt Poincaré's inequality. Clearly, after this, Proposition 1.4.2 only depends on the nonlinearity.

Theorem 5.1.2. *Let $(G_\alpha)_{\alpha\in E}$ with $E = E_1$ (resp. $E = E_2$) be groups acting on $H^1(\Omega)$ in such a way that, for every $g \in G_\alpha$ and for every $u \in H^1(\Omega)$,*

$$g(E) = E, \quad g(E^\perp) = E^\perp, \quad g\alpha = \alpha \quad \text{ and } \mathscr{E}_p(gu) = \mathscr{E}_p(u).$$

Then, for all $M > 0$, if p is close enough to 2, any ground state (resp. least energy nodal) solution $u_p \in \{u \in B(0,M) : P_E(u) \notin B(0, \frac{1}{M})\}$ of Problem $(N\mathscr{P}_p)$ belongs to the invariant set of G_{α_p} where $\alpha_p := P_E u_p$.

We can obtain a little bit more. We show that ground state solution is constant for p close to 2, at least on regular domains.

Theorem 5.1.3. *If Ω is of class $\mathscr{C}^2$ or a product of intervals, there exists $\bar{p} > 2$ such that if $2 < p \leqslant \bar{p}$, then the non-zero constant solutions $\pm\left(\frac{1}{\lambda}\right)^{\frac{1}{p-2}}$ are the unique ground state solutions of Problem $(N\mathscr{P}_p)$.*

Proof. Let us consider a family of positive ground state solutions $(u_p)_{p>2}$. Ideas are similar for negative solutions. By homogeneity, we can w.l.o.g. assume that $\lambda = \lambda_1$. Let us denote by v_p and w_p the projections of u_p respectively on E_1 and $E_1^\perp$. By using the spectral theory, the fact that, as E_1 is the set of constant functions, $|v_p|^{p-1} \in E_1$ and the mean value theorem,

$$\begin{aligned}\lambda_2 \int_\Omega w_p^2 &\le \int_\Omega |\nabla w_p|^2 + w_p^2 \\ &= \lambda \int_\Omega |u_p|^{p-1} w_p \\ &= \lambda \int_\Omega \left((v_p + w_p)^{p-1} - v_p^{p-1}\right) w_p \\ &= \lambda \int_\Omega (p-1)(v_p + \theta_p w_p)^{p-2} w_p^2 \\ &\le \lambda (p-1)(|v_p| + \|w_p\|_\infty)^{p-2} \int_\Omega w_p^2,\end{aligned}$$

for some function θ_p defined from Ω to $(0,1)$. Let us remark that we used the fact that u_p is one-signed in the two first equalities.

Since $\lambda = \lambda_1$, u_p converges uniformly to a non-zero constant (see Proposition 2.1.4). So, we conclude that $(|v_p| + \|w_p\|_\infty)_{p>2}$ is bounded. As $\lambda = \lambda_1 < \lambda_2$, for p close to 2, we have $w_p = 0$. Thus, u_p is a non-zero positive constant. □

Remark 5.1.4. For $\lambda = \lambda_1 = 1$, we obtain ± 1 as non-zero constant solutions.

In fact, we can conclude that constants must even be the unique one-signed solutions for p close to 2.

Theorem 5.1.5. *There exists $\bar{p} > 2$ such that if $2 < p \leqslant \bar{p}$, then the non-zero constant solutions $\pm \left(\frac{1}{\lambda}\right)^{\frac{1}{p-2}}$ are the unique one-signed solutions of Problem ($N\mathscr{P}_p$).*

Proof. Let us consider a family of positive solutions $(u_p)_{p>2}$. Ideas are similar for negative solutions. As in Theorem 5.1.3, we have that

$$\lambda_2 \int_\Omega w_p^2 \le \int_\Omega |\nabla w_p|^2 + w_p^2 \le \lambda_1 (p-1)(2\|u_p\|_\infty)^{p-2} \int_\Omega w_p^2.$$

To conclude the result, it is enough to prove that $\|u_p\|_\infty$ is bounded. For this, as $\|u_p\|_\infty \le c\|u_p\|^{n(p-1)}$ for some $c > 0$ and $n \in \mathbb{N}$ independent of p (see Remark 2.1.5), we prove that $\|u_p\|$ is bounded. On one hand, an integration of the equation leads to $\int_\Omega u_p = \int_\Omega u_p^{p-1}$. So, letting $r = 1$ and $s = p-1$ in the inequality $\|u\|_r \le |\Omega|^{\frac{1}{r} - \frac{1}{s}} \|u\|_s$, we have $\int_\Omega u_p \le |\Omega|$. On the other hand, by using a result of W. M. Ni and Y. Takagi [69], $\|u_p\|$ can be continuously controled by $\int_\Omega u_p^{p-1}$. Moreover, the constant appearing in the inequality is bounded with respect to p. The idea is the use of a bootstrap. As $\int_\Omega u_p^{p-1}$ is bounded, u_p is bounded in $W^{2,1}(\Omega)$ (see Theorem 2.1.1 or [2]). So, we conclude directly in dimension 2 as $W^{2,1}(\Omega)$ can be embedded in $H^1(\Omega)$.

For larger dimensions, as $W^{2,1}(\Omega)$ can be embedded in $L^{N/(N-2)}(\Omega)$, we obtain a control of u_p in $W^{2,\frac{N}{(N-2)(p-1)}}(\Omega)$ and so on until the exponent is high enough for the space to be embedded in $H^1(\Omega)$. □

Corollary 5.1.6. *When $\lambda = \lambda_1$, any family $(u_p)_{2<p\le\bar{p}}$ of one-signed solutions for the problem $(N\mathscr{P}_p)$ is bounded in $H^1(\Omega)$.*

5.1.3 Symmetry breaking of least energy nodal solutions on rectangles

On squares, second eigenfunctions of $-\Delta + \mathrm{id}$ with NBC possess the same kind of symmetry as for DBC. In particular, on $(-1,1)^2$, E_2 is generated by

$$v_1(x,y) = \sin\left(\frac{\pi}{2}x\right) \quad \text{and} \quad v_2(x,y) = \sin\left(\frac{\pi}{2}y\right).$$

So, as the reduced functional $\mathscr{E}_*$ has the same structure as for DBC (see Figure 5.1), we can deduce the same kind of symmetry breaking as for Theorem 1.5.4.

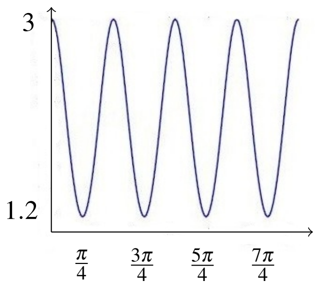


Figure 5.1: Computation of S_* for the Lane–Emden Problem with NBC.

Theorem 5.1.7. *There exist some rectangles such that the least energy nodal solutions of Problem* ($N\mathscr{P}_p$) *are neither symmetric nor antisymmetric with respect to one of the medians of the rectangles.*

5.1.4 Examples

The following examples illustrate the previous sections. Let us remark that concerning the ground state solutions, MPA has always given as approximation the constant solution for p close to 2. It is consistent with Proposition 5.1.3. So, we do not consider it in our figures and tables. To display some symmetry breaking, unless stated otherwise, we will work with the square $\Omega = (-1,1)^2$ in $\mathbb{R}^2$. Let us recall that

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 1 + \frac{\pi^2}{4}.$$

The first eigenspace E_1 is given by the constant functions.

Lane–Emden Problem As first example, we study Problem $-\Delta u + u = \lambda_2 u^3$ with NBC. Figure 5.1 and Table 5.2 show a nodal numerical solution. The nodal line seems to be a diagonal, as expected on the entropy graph 5.1. We represent the level curves -1, 0 and 1.

	Initial function	$\min u$	$\max u$	$\mathscr{E}_4(u^+)$	$\mathscr{E}_4(u^-)$
λ_2	$\cos(\pi(y+1))$	-1.57	1.55	1.24	1.24

Table 5.1: Characteristics of an approximate l.e.n.s on a square.

Concerning the symmetry breaking, we work with the rectangle of side-lengths 1 and 1.2. Figure 5.3 shows a nodal solution u of the problem $-\Delta u + u = u^3$ with Neumann boundary conditions approximated by the MMPA. We represent the level curves -1, 0 and 1.

To conclude this first example, we give at the reader an idea about the structure of a least energy nodal solution with NBC on different domains. We consider a radial domain and a rectangle (see Figure 5.4). Let us just remark that

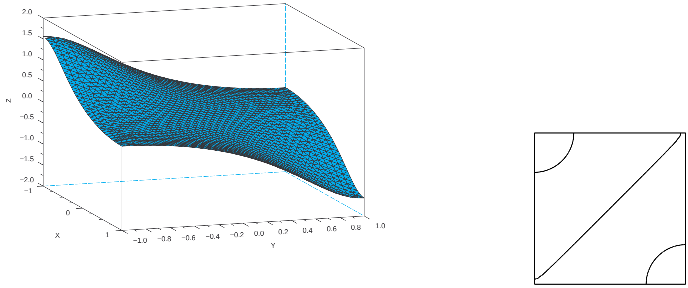


Figure 5.2: Numerical solutions of the Lane–Emden Problem with NBC.

Initial function	$\min u$	$\max u$	$\mathscr{E}_4(u^+)$	$\mathscr{E}_4(u^-)$
$\cos(\pi \frac{x}{2})$	-4.6	6.8	8.2	3.2

Table 5.2: Characteristics of a l.e.n.s. on $(0,1)\times(0,1.2)$.

the same structure will take place in the following examples. In particular, on the rectangle, the solution seems only to have one dimension of variation.

In previous sections, to simplify technical issues, we focused on a power as nonlinearity. However, we can clearly mix the results of Chapter 4 and this one. In this way, we obtain that asymptotic and symmetry results are still valid for more general nonlinearities f_p (satisfying the assumptions of Chapter 4). We just need to be careful with Proposition 5.1.3. To obtain the result, we must be able to control $f_p(v_p+w_p)-f_p(v_p)$ to apply the mean value theorem.

Superlinear perturbation of Lane–Emden Problem Let us consider the following problem

$$\begin{cases} -\Delta u + u = \lambda_2 u^3 + 2u^5, & \text{in } \Omega,\\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

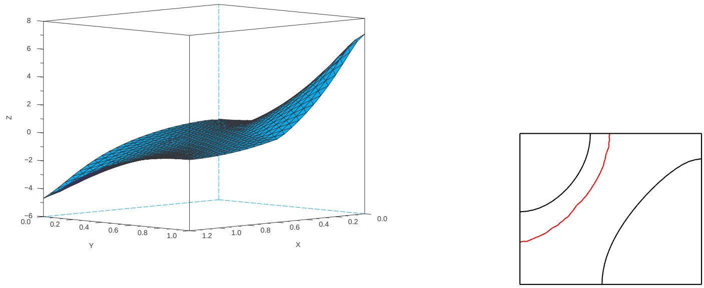


Figure 5.3: Non-symmetric l.e.n.s. for the Lane–Emden Problem with NBC.

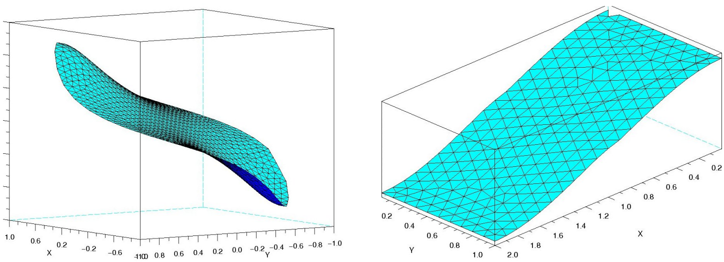


Figure 5.4: Numerical solutions of the Lane–Emden Problem with NBC.

Figure 5.5 and Table 5.3 show nodal numerical solution. The nodal line seems to be a diagonal of the square.

	Initial function	$\min u$	$\max u$	$\mathscr{E}(u^+)$	$\mathscr{E}(u^-)$
λ_2	$\cos(\pi(y+1))$	-1.31	1.33	0.78	0.77

Table 5.3: Characteristics of a l.e.n.s. for a superlinear perturbation with NBC.

Minimizers of $\mathscr{E}_*$ on $\mathscr{N}_*$ seem to be symmetric with respect to a diagonal. Indeed, any second eigenfunction can be written as a multiple of $v_\alpha = \cos(\alpha)v_1 + \sin(\alpha)v_2$, for $\alpha \in [0, 2\pi]$, where v_1 and v_2 have been defined previ-

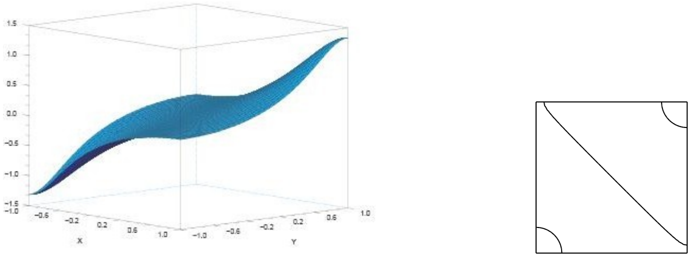


Figure 5.5: L.E.N.S. for a superlinear perturbation with NBC.

ously. Figure 5.6 is a numerical evaluation of the function

$$S_* : [0, 2\pi] \to \mathbb{R} : \alpha \mapsto \mathscr{E}_*(t_\alpha v_\alpha),$$

where t_α is the unique positive scalar such that $v_\alpha \in \mathscr{N}_*$ and $\mathscr{E}_*$ is defined page 96. As the minimum seems to be attained for $\alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$, the corresponding minimizers should be symmetric functions with respect to a diagonal. As a consequence of the previous computation, the symmetry breaking occurs in this example.

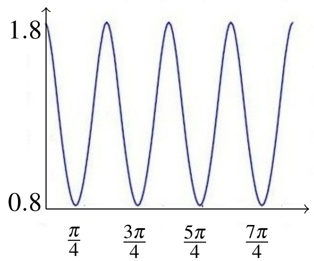


Figure 5.6: Computation of S_* for the superlinear perturbation with NBC.

Exponential growth We consider the problem

$$\begin{cases} -\Delta u + u = \lambda_2 u(e^{u^2} - 1)^{0.5}, & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

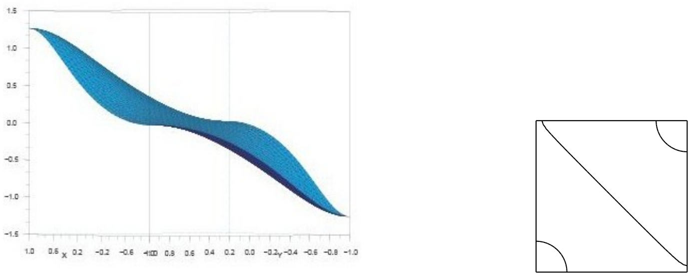


Figure 5.7: L.E.N.S. for an exponential growth with NBC.

	Initial function	$\min u$	$\max u$	$\mathscr{E}_{2.5}(u^+)$	$\mathscr{E}_{2.5}(u^-)$
λ_2	$\cos(\pi(y+1))$	-1.25	1.26	0.9	0.91

Table 5.4: Characteristics of the "exponential" approximate solution with NBC.

Figure 5.7 and Table 5.4 show the results of the numerical experiments. The same comments as in the previous example can be made. In particular, in the case of least energy nodal solutions, minimizers of $\mathscr{E}_*$ on $\mathscr{N}_*$ seem to be symmetric with respect to a diagonal as shown in Figure 5.8 and the symmetry breaking occurs in this example.

Sum of powers We work with the problem

$$\begin{cases} -\Delta u + u = \lambda_2 u \left(\frac{1}{4}|u|^{0.25} + \frac{1}{4}|u|^{0.5} + \frac{1}{2}|u|\right), & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

Figure 5.9 and Table 5.5 show the results of the numerical experiments. We can do the same comments as in the previous cases.

Since the limit functional for this example is the same as the limit functional for the Lane–Emden Problem ($\mathscr{P}_p$), the symmetry breaking phenomenum occurs.

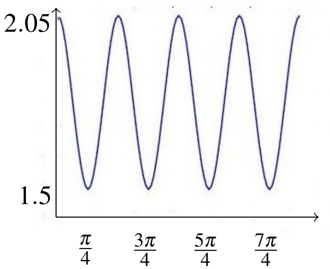


Figure 5.8: Computation of S_* for an exponential growth with NBC.

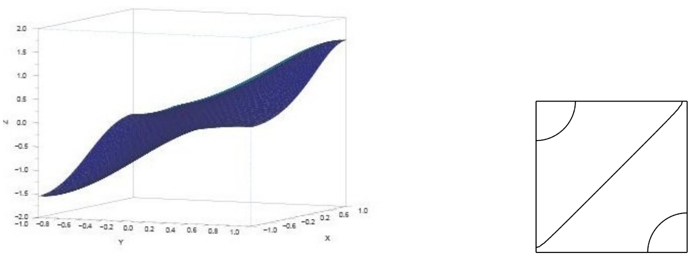


Figure 5.9: L.E.N.S. for a sum of powers with NBC

	Initial Function	$\min u$	$\max u$	$\mathscr{E}_1(u^+)$	$\mathscr{E}_1(u^-)$
λ_2	$\cos(\pi y)$	-1.55	1.55	0.4	0.5

Table 5.5: Characteristics of the “sum of powers” approximate solution.

5.2 Ground state solutions for p large

We know that the first eigenvalue λ_1 of the operator $-\Delta + \mathrm{id}$ is 1. So, as we do not work with least energy nodal solutions in this section, by homogeneity, we can avoid to mention λ in the problems studied. By remark 5.1.4, non-zero constant solutions are ± 1.

5.2.1 The ground state solutions are not radial for p large

As mentioned in the Introduction, we expect a symmetry breaking. To know for which p a ground state solution of Problem ($N\mathscr{P}_p$) is not ± 1 anymore, we will use Morse index of $\mathscr{E}_p$ at ± 1. Let us consider the linearized problem of ($N\mathscr{P}_p$) around $u \in H^1(\Omega)$,

$$\begin{cases} -\Delta h + h - (p-1)|u|^{p-2}h = \mu h, & \text{in } \Omega, \\ \partial_\nu h = 0, & \text{on } \partial\Omega. \end{cases} \tag{$N\mathscr{P}'_p$}$$

Proposition 5.2.1. *For any $i \in \mathbb{N} \setminus \{0\}$, if $\lambda_i < p-1 \leq \lambda_{i+1}$ then $\operatorname{ind}_M(\pm 1, \mathscr{E}_p)$ equals $\sum_{k \leq i} \dim E_k$.*

Proof. By definition, the Morse index of $\mathscr{E}_p$ at a critical point u corresponds to the sum of the dimensions of the eigenspaces associated to the negative eigenvalues μ of the Problem ($N\mathscr{P}'_p$).

Let us consider $u = 1$ or -1. The solutions h of Problem ($N\mathscr{P}'_p$) are eigenfunctions of $-\Delta + \mathrm{id}$ associated to the eigenvalue $p-1+\mu$. We obtain that $\mu_i = \lambda_i - (p-1)$, which concludes the proof. □

As a ground state solution has a Morse index equals to 1, we obtain the following corollary.

Corollary 5.2.2. *Function 1 (resp. -1) cannot be a ground state solution if*

$$p - 1 > \lambda_2.$$

5.2.2 Bifurcation results

In this section, we try to obtain some bifurcations of solutions starting from ± 1. By using previous Krasnoselskii-Boehme-Marino bifurcation result (see page 120), we obtain that the first bifurcation point $(p, \pm 1)$ occurs for $p = 1 + \lambda_2$.

Theorem 5.2.3. *For $2 < p < 2^*$, $(p, \pm 1)$ is a bifurcation point for the problem ($N\mathscr{P}_p$) if and only if $p = 1 + \lambda_i$, for $i > 1$.*

Proof. We prove it for 1. The idea is similar for -1. First, we rephrase our Problem ($N\mathscr{P}_p$). The fact that (p, u) is a solution of Problem ($N\mathscr{P}_p$) is equivalent to the fact that $(p-1, u-1) = (q, v)$ solves $L^*(q, v) - N^*(q, v) = 0$ where

$$L^* : (1, 2^*-1) \times H^1 \to (H^1)' : (q,v) \mapsto (-\Delta + \mathrm{id})v - qv,$$
$$N^* : (1, 2^*-1) \times H^1 \to (H^1)' : (q,v) \mapsto |1+v|^{q-1}(1+v) - qv - 1.$$

For technical reasons, we work with equation $q^{-1}(L^* - N^*) = 0$. We use the third part of the Krasnoselskii-Boehme-Marino result. Composing with the canonical isomorphism between $H^1(\Omega)$ and $(H^1(\Omega))'$, $q^{-1}L^*(q,v)$ equals $q^{-1}v - (-\Delta + \mathrm{id})^{-1}v$ in $H^1(\Omega)$. This plays the role of operator L in equation (5.4). It is enough to prove that $\lim_{v\to 0} \frac{N^*(q,v)}{\|v\|} = 0$ uniformly on compact sets of q in $(1, 2^*-1)$. It will be established if we show that if $(v_n)_{n\in\mathbb{N}}$ is a sequence in $H^1(\Omega)$ with norm equals to 1, $(t_n)_{n\in\mathbb{N}}$ converging to 0 and $(q_n)_{n\in\mathbb{N}}$ bounded by q, then, up to a subsequence, $\lim_{n\to\infty} \frac{N^*(q_n, t_n v_n)}{t_n} = 0$. We show that

$$\lim_{n\to\infty} \frac{1}{t_n} \int_\Omega \left(|1+t_n v_n|^{q_n-1}(1+t_n v_n) - 1 - q_n t_n v_n\right) w = 0,$$

uniformly with respect to $w \in H^1(\Omega)$ with $\|w\| = 1$. We have, for any $n \in \mathbb{N}$, because $\frac{q_n+1}{q_n}$ and q_n+1 are less than 2^*,

$$\begin{aligned}
&\frac{1}{t_n} \int_\Omega \left(|1+t_n v_n|^{q_n-1}(1+t_n v_n) - 1 - q_n t_n v_n\right) w \\
&\qquad \le \frac{1}{t_n} \left\| |1+t_n v_n|^{q_n-1}(1+t_n v_n) - 1 - q_n t_n v_n \right\|_{\frac{q_n+1}{q_n}} \|w\|_{q_n+1}.
\end{aligned}$$

By using the mean value theorem, there exists $\theta_n \in (0,1)$ such that

$$\frac{1}{t_n}\left(|1+t_n v_n|^{q_n-1}(1+t_n v_n) - 1 - q_n t_n v_n\right) = q_n |1+\theta_n t_n v_n|^{q_n-1} v_n - q_n v_n \to 0$$

almost everywhere and, as we can assume that $0 \le t_n \le 1$ w.l.o.g.,

$$\left|\frac{1}{t_n}\left(\left|1+t_n v_n\right|^{q_n-1}(1+t_n v_n) - 1 - q_n t_n v_n\right)\right| \le 2q_n \left|1+|v_n|\right|^q \le 2q\left|1+|v_n|\right|^q. \tag{5.6}$$

Using Lebesgue's dominated convergence theorem, we infer the expected result.

Thus, as the Problem $(N\mathscr{P}_p)$ respects the Palais-Smale condition, we can apply the third point of the Krasnoselskii-Boehme-Marino result (see page 120) to conclude the existence of bifurcation branches starting from 0 when q equals λ_i. So, our Problem $(N\mathscr{P}_p)$ has bifurcations starting from 1 when p equals $1+\lambda_i$, for some i.

If $p-1 \neq \lambda_i$, we obtain that there does not exist bifurcations. Indeed, the operator $\partial_v(L^* - N^*)(p-1,0)$ is invertible. We conclude by the implicit function theorem (see page 34). □

Remark 5.2.4.
- Using the same arguments, we can conclude from the second statement of the Krasnoselskii-Boehme-Marino result (see page 120) that if λ_i has odd multiplicity then we obtain global bifurcations.
- By Rabinowitz's principle (see page 121), we obtain that bifurcation branches are unbounded or linked together by pair when the multiplicity of λ_i is odd.
- In dimension $N = 2$, as $2^* = +\infty$, we have infinitely many bifurcations.

5.2.3 Specific case: a ball

In this section, we focus on the ball $B(0,1)$. We already know that a ground state solution cannot be a constant function for $p > 1 + \lambda_2$ (see Corollary 5.2.2). We would like to know whether they stay radial functions.

Non-radial solution We start by adapting to the Neumann boundary case, results of A. Aftalion and F. Pacella (see [3]). They proved, for the Lane-Emden Problem with Dirichlet boundary conditions, that nodal radial solutions has a Morse index at least equals to $N+1$.

Lemma 5.2.5. *Let $Lv := -\Delta v + v - (p-1)|u_p|^{p-2}v$, u_p a solution of Problem $(N\mathscr{P}_p)$. Let us denote, for any $i \in \{1,2,\dots,N\}$,*

$$\Omega_i = \{x = (x_1,\dots,x_N) \in B(0,1) : x_i = 0\},$$

$$\Omega_i^+ = \{x = (x_1,\dots,x_N) \in B(0,1) : x_i > 0\}$$

and

$$\Omega_i^- = \{x = (x_1,\dots,x_N) \in B(0,1) : x_i < 0\}.$$

Let μ_i denote the first eigenvalue of L in Ω_i^+ with Dirichlet boundary conditions on Ω_i and Neumann boundary conditions on $\partial\Omega_i^+ \setminus \Omega_i$; and ψ_i the first eigenfunction related to μ_i. If u_p is even with respect to x_i, let us denote ψ_i^* the *extension of ψ_i on all $B(0,1)$ such that ψ_i^* is odd with respect to x_i. Then, ψ_i^* is not a first eigenfunction of L in $B(0,1)$ with Neumann boundary conditions.*

Moreover, if u_p is even with respect to the variables $x_1,\dots,x_k$, $1 \le k \le N$, $\psi_1^*,\dots,\psi_k^*$ constructed are independent and give k eigenfunctions of L (none of which is a first eigenfunction).

Proof. As the potential $1-(p-1)|u_p|^{p-2}$ is bounded, the first eigenvalue of L is real and the unique first eigenfunction is a one-signed function (see e.g. [34]). We clearly have $L(\psi_i^*) = \mu_i \psi_i^*$ on $B(0,1)\setminus\Omega_i$ and ψ_i^* satisfies the Neumann boundary conditions on $\partial B(0,1)$. It remains to verify that $L(\psi_i^*) = \mu_i\psi_i^*$ on Ω_i. As $\psi_i^* = 0$ on Ω_i, we have

$$\forall j \neq i,\ \partial_{x_j}\psi_i^* = \partial_{x_j}^2\psi_i^* = 0.$$

As ψ_i is an eigenfunction of L, by a continuous extension of $\partial_{x_i}^2\psi_i$ on Ω_i and as ψ_i^* is odd, we have

$$-\sum_{j=0}^{N}\partial_{x_j}^2\psi + 0 - (p-1)|u_p|^{p-2}0 = 0$$

and thus $\partial_{x_i}^2\psi_i^* = 0$ on Ω_i. So, $L(\psi_i^*) = \mu_i\psi_i^*$ on Ω_i. As ψ_i^* is a sign-changing function, it is not a first eigenfuntion. It concludes the first part of the proof.

The independence of ψ_i^*, for $i \in \{1,\dots,k\}$, is easily established using the fact that ψ_i^* is the only function which is not equal to zero in the direction of x_i. □

Proposition 5.2.6. *If u_p is a non-constant one-signed radial solution, its Morse index is at least $N+1$.*

Proof. Without loss of generality, we can assume that $u_p \ge 0$ on $B(0,1)$. By Lemma 5.2.5, we have the existence of N independent eigenfunctions ψ_i^* (none of which is a first eigenfunction), for $i \in \{1,\dots,N\}$, of $Lv := -\Delta v + v - (p-1)|u|^{p-1}v$ with Neumann boundary conditions. Since the corresponding eigenvalues of ψ_i^* are above the first one, it is sufficient to prove that eigenvalues related to ψ_i^* are negative.

Let $i \in \{1,\dots,N\}$. We have $\partial_{x_i}u_p = 0$ on Ω_i (because u_p is radial) and on $\partial B(0,1)$ (because u_p is constant on the boundary and satisfies the Neumann boundary conditions). Pick $\bar{x} \in \Omega_i^+$ such that $\partial_{x_i} u_p(\bar{x}) \neq 0$ (such $\bar{x}$ exists because u_p is radially symmetric and not constant). Let D be the connected component of $\{x | \partial_{x_i} u_p(x) \neq 0\}$ containing $\bar{x}$.

Differentiation of the equation ($N\mathscr{P}_p$) for which u_p is a solution gives

$$L(\partial_{x_i} u_p) = -\Delta(\partial_{x_i} u_p) + \partial_{x_i} u_p - (p-1)|u_p|^{p-2} \partial_{x_i} u = 0.$$

Since $\partial_{x_i} u_p$ does not change sign on D, 0 is the first eigenvalue of L in D with Dirichlet boundary conditions. As $D \subseteq \Omega_i^+$, the first eigenvalue of L in Ω_i^+ for Dirichlet boundary conditions is non-positive. By applying the Höpf's Lemma (see 2.2.1), we know that first eigenvalue for mixed boundary conditions is strictly lower than first eigenvalue for Dirichlet boundary conditions. So, we have that the first eigenvalue of L in Ω_i^+ for Dirichlet boundary conditions on Ω_i and Neumann on $\partial B(0,1) \setminus \Omega_i$ is negative, which concludes the proof. □

As the Morse index of ground state solutions is 1, the previous result implies the following theorem.

Theorem 5.2.7. *If u_p is a radially symmetric ground state solution, then u_p is constant.*

Study of eigenspaces Existence of radial bifurcations proved in the following will be related to eigenvalues λ_i and eigenspaces E_i of $-\Delta + \mathrm{id}$ in $H^1(B(0,1))$ with Neumann boundary conditions. Let us explain here the structure of these functions. The technique is the same as for Dirichlet boundary conditions (see Section 1.3 in Chapter 1). The first eigenvalue λ_1 equals 1 and first eigenfunctions are the constant functions. By the separation of variable $r = |x|$ and $\sigma = \frac{x}{|x|}$, for each $i > 1$, there exists an unique $(k,q) \in \mathbb{N} \times \mathbb{N} \setminus \{0\}$ such that, for all eigenfunction u associated to λ_i,

$\lambda_i = j_{\mu_k,q}^2 + 1$ with $j_{\mu_k,q}$ the q^{st} positive critical point of the Bessel function J_{μ_k},

where $\mu_k = \left(k + \frac{N-2}{2}\right)$. We have that

$$u(x) = J_{\mu_k}(j_{\mu_k,q} r) r^{\frac{2-N}{2}} P_k\left(\frac{x}{|x|}\right) \text{ with}$$

$$P_k : \mathbb{R}^N \to \mathbb{R} \text{ an harmonic homogenous polynomial of degree } k.$$

Let us remark that radial eigenfunctions are given when $k = 0$, i.e. when we consider critical points of the Bessel function $J_{\frac{N-2}{2}}$. For each eigenspace E_i, the dimension of the intersection with radial functions is 0 or 1. Moreover, for functions in E_i with $k = 0$, the radial part is given by $J_{\mu_0}(j_{\mu_0,q}r)r^{\frac{2-N}{2}}$. We remark that intersections of those functions with 0 are simple.

Radial bifurcation We would like to know whether solutions along the bifurcations given in Proposition 5.2.3 belong to $\mathrm{H}^1_{\mathrm{rad}}(B(0,1))$, the subspace of radially symmetric functions of $H^1(B(0,1))$. In particular, we obtain that bifurcation branches at $(1+\lambda_2, \pm 1)$ are not radial.

Theorem 5.2.8.
- *If* $E_i \cap \mathrm{H}^1_{\mathrm{rad}}(B(0,1)) = \{0\}$, $(1+\lambda_i, \pm 1)$ *is not a radial bifurcation point in* $\mathrm{H}^1_{\mathrm{rad}}(B(0,1))$.
- *If* $E_i \cap \mathrm{H}^1_{\mathrm{rad}}(B(0,1)) \neq \{0\}$, $(1+\lambda_i, \pm 1)$ *is a radial bifurcation point in* $\mathrm{H}^1_{\mathrm{rad}}(B(0,1))$.

Proof. We consider the operator $L^* - N^*$ defined in the proof of Theorem 5.2.3 restricted to $\mathrm{H}^1_{\mathrm{rad}}(B(0,1))$. If the eigenspace E_i of $-\Delta + \mathrm{id}$ does not contain non-trivial radial functions, the operator

$$\partial_u (L^* - N^*)_{|\mathrm{H}^1_{\mathrm{rad}}}(\lambda_i, 0)$$

is invertible. Indeed, the operator is a Fredholm operator of index 0 and $\partial_u (L^* - N^*)_{|\mathrm{H}^1_{\mathrm{rad}}}(\lambda_i, 0) w = 0$ if and only if

$$w \in E_i \cap \mathrm{H}^1_{\mathrm{rad}}(B(0,1)).$$

So, by the first point of the Krasnoselskii-Boehme-Marino result (see page 120), $(\lambda_i, 0)$ is not a bifurcation point for $(L^* - N^*)_{|\mathrm{H}^1_{\mathrm{rad}}}$.

Now, if $E_i \cap \mathrm{H}^1_{\mathrm{rad}}(B(0,1)) \neq \{0\}$, we can follow the same arguments as in the proof of Theorem 5.2.3 in the space $\mathrm{H}^1_{\mathrm{rad}}$ to establish the existence of a radial bifurcation point. □

Remark 5.2.9. Let us consider the ball of radius R. When $R \to +\infty$, we obtain that eigenvalues λ_i of $-\Delta + \mathrm{id}$ go to 1. So, at $2 < p < 2^*$ fixed, for any $n > 0$, there exists $R_n > 1$ such that, on the ball B_{R_n}, we have at least n radial global bifurcations.

5.2.4 Numerical symmetry breaking and conjecture

As ± 1 are not ground state solutions for $p > 1 + \lambda_2$ and as the first bifurcation is given when $p = 1 + \lambda_2$, we think that the solutions on this bifurcation gives the new branch of ground state solutions. Here, we study it numerically to make a conjecture.

Let us recall that, in dimension $N = 2$, on the unit ball,

$$\lambda_1(B(0,1)) = 1 \text{ and } \lambda_2(B(0,1)) = 2 + \frac{\pi^2}{4}.$$

The second eigenspace E_2 has an even multiplicity and the non-zero second eigenfunctions are not radial.

Let us consider the following problems

$$\begin{cases} -\Delta u + u = |u|^{1.1+\frac{\pi^2}{4}} u, & \text{in } B(0,1), \\ \partial_\nu u = 0, & \text{on } \partial B(0,1). \end{cases} \tag{5.7}$$

Figure 5.10 and Table 5.6 show the result of the numerical experiment. As expected, it is not the function 1, the energy is strictly inferior as the energy of 1, it is positive and not a radial function (but seems to be Schwarz foliated).

$\min u$	$\max u$	$\mathscr{E}(u)$	$\mathscr{E}(1)$
0.8	1.2	0.98	1.0

Table 5.6: Characteristics of the non-symmetric ground state solution for NBC.

Numerical results permit us to make the next conjecture.

Conjecture 5.2.10. On a ball, in dimension $N = 2$, there is a non-radial symmetry breaking of ground state solutions when p crosses $1 + \lambda_2$.

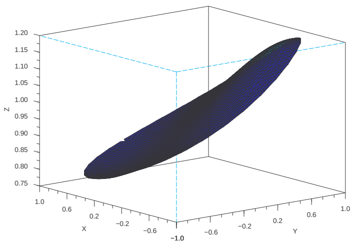


Figure 5.10: Non-symmetric ground state solution for NBC.

Chapter 6

General potentials: mountain pass algorithms

In this last chapter, we consider the problem

$$\begin{cases} -\Delta u(x) + V(x)u(x) = f(u(x)), & x \in \Omega, \\ \qquad\qquad\qquad u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{SP}$$

defined on an open bounded connected domain Ω *in* $\mathbb{R}^N$ *and* $N \geq 2$*. We also pay some attention to the case* $\Omega = \mathbb{R}^N$*. Let us remark that, this time, the nonlinearity* f *is fixed and do not depend on a parameter* p*.*

About the operator $-\Delta + V$*, we assume that*

(V_1) V *is continuous on* $\overline{\Omega}$ *and* 0 *does not belong to the spectrum* $\sigma(-\Delta + V)$*.*

Let us remark that we accept that the linear operator $-\Delta + V$ *possesses negative eigenvalues. As zero does not belong to the spectrum of* $-\Delta + V$*, we have the decomposition* $H_0^1(\Omega) = H^{(+)} \oplus H^{(-)}$ *corresponding to the positive and negative eigenspaces of* $-\Delta + V$*. For any* $u \in H_0^1(\Omega)$*, we can uniquely decompose* $u = u^{(+)} + u^{(-)}$ *in* $H^{(+)} \oplus H^{(-)}$ *such that*

$$\int_\Omega |\nabla u|^2 + Vu^2 = \int_\Omega |\nabla u^{(+)}|^2 + V(u^{(+)})^2 + \int_\Omega |\nabla u^{(-)}|^2 + V(u^{(-)})^2$$

with $\int_\Omega |\nabla u^{(+)}|^2 + V(u^{(+)})^2 \geq 0$ *and* $\int_\Omega |\nabla u^{(-)}|^2 + V(u^{(-)})^2 \leq 0$*.*

About f*, we assume that*

(F_1) *f is continuous and subcritical, i.e. there exist $a > 0$ and $p \in (2, 2^*)$ such that, for any $t \in \mathbb{R}$, $|f(t)| \leq a(1 + |t|^{p-1})$;*

(F_2) $\lim_{|t|\to 0} \frac{f(t)}{t} = 0$;

(F_3) $\lim_{|t|\to +\infty} \frac{F(t)}{t^2} \to +\infty$;

(F_4) *$t \mapsto \frac{f(t)}{|t|}$ is increasing on $(-\infty, 0)$ and $(0, +\infty)$,*

where $F(t) := \int_0^t f(s)\,\mathrm{d}s$.

Solutions of Problem (SP) are the critical points of the energy functional

$$\mathscr{E} : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega \left(|\nabla u(x)|^2 + V(x)u(x)^2\right) \mathrm{d}x - \int_\Omega F(u(x))\,\mathrm{d}x. \quad (6.1)$$

Let us point that the interesting feature of this problem is the fact that zero is not required to be a local minimum of $\mathscr{E}$ (cfr. directions in $H^{(-)}$ when $H^{(-)} \neq \{0\}$). In this case, the mountain pass geometry does not hold anymore (see Figure 1.1).

Recently, in [64], A. Szulkin and T. Weth proved that this problem possesses a non-trivial solution with minimum energy, i.e. a ground state solution. Let us just remark that they do not need the usual super-quadraticity assumption (SQ). So, they generalize classical results of P. H. Rabinowitz (see e.g. [61]) where (SQ) was required.

(SQ) $\exists a > 2,\ \forall u \in \mathbb{R} \setminus \{0\},\ aF(u) \geq f(u)u > 0.$

Moreover, they prove the existence of a ground state solution if one works with $\Omega = \mathbb{R}^N$. One just needs to require in more some periodicity assumptions on f and V:

(PA) *f and V are 1-periodic in x_i, for $i = 1, \dots, N$.*

To obtain it, inspired by a result of A. Pankov [56], they introduce a variant of the classical Nehari manifold defined previously: a ground state solution achieves the minimum of the energy functional $\mathscr{E}$ on the generalized Nehari manifold $\mathscr{N}_G$

$$\mathscr{N}_G := \left\{u \in H_0^1(\Omega) \setminus \{0\} : \langle \mathrm{d}\mathscr{E}(u), v\rangle = 0, \text{ for any } v \in H^{(-)}\right\}. \quad (6.2)$$

So, in some sense, they remove "descent" directions in $H^{(-)}$. Moreover, they prove the following properties.

1. *The minimum of the energy $\mathscr{E}$ on $\mathscr{N}_G$ is attained by a non-zero function and is a critical point of $\mathscr{E}$.*

2. *For each $u \in H_0^1(\Omega) \setminus H^{(-)}$, $\mathscr{N}_G$ intersects*

$$H^*(u) := \mathbb{R}^{>0} u \oplus H^{(-)} = \mathbb{R}^{>0} u^{(+)} \oplus H^{(-)}$$

   in one and only one function which is the unique point at which the functional $\mathscr{E}$ achieves its maximum on $H^*(u)$.

3. *The function*

$$P : H_0^1(\Omega) \setminus H^{(-)} \to \mathscr{N}_G : u \mapsto P(u)$$

   such that $\mathscr{E}$ is maximum on $H^*(u)$ at $P(u)$ defines a continuous projection on $\mathscr{N}_G$. Restricted to the unit sphere S^+ of $H^{(+)}$, $P_{|S^+}$ defines an homeomorphism between S^+ and $\mathscr{N}_G$.

Let us just remark that techniques defined in Chapter 1 *to study the asymptotic behavior and symmetries are not working when $H^{(-)}$ is not the trivial space $\{0\}$. As we do not know the exact coefficients of the projection P on the generalized Nehari manifold anymore, results 1.1.2 and 1.1.6 concerning a priori bounds cannot be used. Here, we are interested in the convergence of a constrained steepest descent algorithm to approach a non-zero solution of Problem (SP), i.e. a critical point of $\mathscr{E}$ on $\mathscr{N}_G$. We pay attention to the ground state solutions and make a conjecture on their symmetries in the case where V is constant and the nonlinearity f is a small power.*

Let us now outline the numerical approach. To start, we recall what happens in the classical case where the mountain pass geometry is respected, i.e. $H^{(-)} = \{0\}$.

***Classical case**: The projection $P_{\mathscr{N}}$ on the classical Nehari manifold $\mathscr{N}$ is just given by*

$$P_{\mathscr{N}}(u) = t_* u,$$

with $\mathscr{E}(t_* u) = \max_{t>0} \mathscr{E}(tu)$. In the former case, Y. Li and J. Zhou (see e.g. [71, 72]) proved that accumulation points of the sequence obtained by the following algorithm (also called mountain pass algorithm) are non-zero solutions of Problem (SP).

Mountain Pass Algorithm (MPA). *1. Choose $u_0 \in \mathscr{N}$ and $n \leftarrow 0$;*

2. *If $\nabla\mathscr{E}(u_n) = 0$, then stop;*
   else, $n \leftarrow n+1$ and compute

$$u_{n+1} = P_{\mathscr{N}}\left(u_n - s_n \frac{\nabla\mathscr{E}(u_n)}{\|\nabla\mathscr{E}(u_n)\|}\right),$$

   where $s_n \in \mathbb{R}^+ \setminus \{0\}$ is defined such that

$$\mathscr{E}(u_{n+1}) - \mathscr{E}(u_n) < -\frac{s_n}{2}\|\nabla\mathscr{E}(u_n)\|$$

   and is sufficiently large (see Section 6.1 for a precise definition of the term "sufficiently");

3. *Go to step 2.*

A priori, the solution may not be a ground state solution. Nevertheless, concerning problems where the ground state solutions are one-signed functions and where the one-signed solutions are unique up to the sign (e.g. the Lane–Emden problem, see [32]), we obtain that if an accumulation point is one-signed then it is a ground state solution a posteriori.

To seek sign-changing solutions, one can define a projection $P_{\mathscr{M}}$ from sign-changing functions in $H_0^1(\Omega)$ to the nodal Nehari set $\mathscr{M}$ (see Introduction of Chapter 1 for a definition) by

$$P_{\mathscr{M}}(u) = P_{\mathscr{N}}(u^+) + P_{\mathscr{N}}(u^-),$$

with $u^+ := \max(u,0)$ and $u^- := \min(u,0)$. If one replaces $P_{\mathscr{N}}$ with $P_{\mathscr{M}}$ in MPA, one obtains the modified mountain pass algorithm (MMPA, see e.g. [25]). In practice, the MMPA seems to converge up to a subsequence to sign-changing critical points of $\mathscr{E}$. Nevertheless, to our knowledge, there is no proof of convergence. In two words, the problem is related to the fact that $\mathscr{M}$ is not a manifold anymore and to the "bad" properties of $P_{\mathscr{M}}$. It will be explained later in detail (see Lemma 6.1.9). An open question is the definition of a "good" projection $P_{\mathscr{M}}$ ensuring a convergence of the constrained steepest descent algorithm. Let us remark that other algorithms have been studied with the aim to approach critical points of $\mathscr{E}$. For example, in 2001, the "high linking algorithm" (see *e.g. [25, 41]) has been introduced: starting from a "MPA" solution, it permits to compute critical points of $\mathscr{E}$ with higher energy but, as for MMPA, the convergence is not established. An other one is a bissection method introduced by V. Barutello and S. Terracini (see e.g. [9, 10]). A proof of the convergence is given but, in practice, the algorithm requires to "know" a priori some levels of $\mathscr{E}$, which is not so easy to obtain.*

***Generalized case**: In the case of Problem (SP) where $H^{(-)}$ may be different from $\{0\}$, we introduce a generalized mountain pass algorithm: we prove that if one replaces $P_{\mathscr{N}}$ with P in the MPA (see page 142), the algorithm is well-defined, "stable" and converges. Moreover, any accumulation point is a solution of Problem (SP). Let us mention that the convergence of the algorithm depends on the structure of the energy functional $\mathscr{E}$ and not directly on Problem (SP). In some sense, the Problem (SP) is just an example. This is why, in Section 6.1, we work with general Hilbert space and general functional $\mathscr{E}$. We give assumptions on $\mathscr{E}$ such that the generalized mountain pass algorithm converges to a non-zero critical point of $\mathscr{E}$. The proof draw its inspiration from the paper [66] written by N. Tacheny and C. Troestler. There, the authors were interested in the convergence up to a subsequence of a mountain pass algorithm under some additional restrictions: they were looking for solutions in some cones of $H_0^1(\Omega)$. An interesting fact of [66] is a precise definition of allowed stepsize s_n in the MPA (see page 142) in such a way one can control in some sense $\mathscr{E}(u_n - s_n \frac{\nabla u_n}{\|\nabla u_n\|})$. This is an important ingredient to obtain the convergence up to a subsequence. Here, we improve the method by defining a new stepsize s_n for the MPA in order to control $\mathscr{E}(u_n - s\frac{\nabla u_n}{\|\nabla u_n\|})$ for any $0 < s < s_n$. It allows us to obtain, in Section 6.1.2 and under a "localization" assumption about a critical point, the convergence of the algorithm. Let us remark that, as ground state solutions can be sign-changing solutions and as we do not have the uniqueness of the solution, we cannot verify anymore a posteriori whether solutions are ground state solutions or not. Nevertheless, a steepest descent method increases the likelyhood that they are. We also study what happens in the case where $\Omega = \mathbb{R}^N$.*

To conclude the chapter, we use the generalized MPA to make a conjecture about the symmetries of ground state solutions if $V = \lambda < -\lambda_1$ is a constant function with λ_i (resp. E_i) the eigenvalues (resp. eigenspaces) of $-\Delta$ in $H_0^1(\Omega)$

with DBC and if $f(x) := |x|^{p-2}x$*. For now, only partial theoretical results for the case where* $-\lambda_2 < \lambda < -\lambda_1$ *(i.e.* $\dim H^{(-)} = 1$*) and* Ω *is a ball are known: if* p *is the critical exponent* 2^**, A. Szulkin, T. Weth and M. Willem obtained that a ground state solution on radial domains is Schwarz foliated symmetric, non-radial and has exactly two nodal domains (see e.g. [65]). In this case, the ground state solutions are at the same time least energy nodal solutions.*

This work is inspired by [37] written in collaboration with C. Troestler.

6.1 Constrained steepest descent method for indefinite problems

Throughout this section, we denote by $\mathscr{H}$ a Hilbert space with inner product $\langle\cdot|\cdot\rangle$ and norm $\|\cdot\|$, E a closed subspace of $\mathscr{H}$ with orthogonal projection P_E, $\mathscr{E} : \mathscr{H} \to \mathbb{R}$ a $\mathscr{C}^1$-functional, and $P : \mathscr{H} \setminus E \to \mathscr{H} \setminus \{0\}$ a "peak selection" (i.e. for any $u \in \mathscr{H} \setminus E$, $P(u)$ is a local maximum of $\mathscr{E}$ on $\mathbb{R}^+ u^* \oplus E$, where $u^* := P_{E^\perp} u = u - P_E(u)$). We would like to define s_n such that the MPA (see page 142) converges to a critical point of $\mathscr{E}$ when we are starting from $u_0 \in \mathrm{Im}(P)$ and we are considering the projection P.

Let us start by proving that the energy functional $\mathscr{E}$ decreases "sufficiently fast" when one follows the negative gradient. This result is classical. Readers can find an example of proof in [66, 71].

Lemma 6.1.1. *If* P *is continuous at* $u_0 \in \mathrm{Im}(P)$*,* $u_0 \notin E$ *and* $\nabla\mathscr{E}(u_0) \neq 0$*, there exists* $s_0 > 0$ *such that, for any* $0 < s < s_0$*,*

$$\mathscr{E}(P(u_s)) - \mathscr{E}(u_0) < -\frac{s}{2}\|\nabla\mathscr{E}(u_0)\|,$$

where

$$u_s := u_0 - s\frac{\nabla\mathscr{E}(u_0)}{\|\nabla\mathscr{E}(u_0)\|}.$$

This deformation Lemma 6.1.1 implies directly the next result.

Theorem 6.1.2. *If* $u_0 \in \mathrm{Im}\,P$ *is a local minimum of* $\mathscr{E}$ *on* $\mathrm{Im}\,P$ *at which* P *is continuous, then* u_0 *is a critical point of* $\mathscr{E}$*.*

Proof. If u_0 were not a critical point of $\mathscr{E}$, i.e. $\nabla\mathscr{E}(u_0) \neq 0$, Lemma 6.1.1 would imply the existence of a $s_0 > 0$ such that, for any $s \in (0, s_0)$,

$$\mathscr{E}(P(u_s)) < \mathscr{E}(u_0) - \frac{s}{2}\|\nabla\mathscr{E}(u_0)\| < \mathscr{E}(u_0),$$

which is, for s small, a contradiction with the definition of local minimum. □

In fact, we can improve the deformation Lemma 6.1.1 and obtain the following uniform deformation Lemma 6.1.3. This result gives a local uniformity around u_0. It will be a key to obtain the convergence of the algorithm.

Lemma 6.1.3. *If P is continuous at $u_0 \in \mathrm{Im}(P)$, $u_0 \notin E$ and $\nabla\mathscr{E}(u_0) \neq 0$, there exist $s_0 > 0$ and $r_0 > 0$ such that, for any $0 < s \leq s_0$ and $u \in B(u_0, r_0)$, $\nabla\mathscr{E}(u) \neq 0$, $u_s \neq 0$ where $u_s := u - s\frac{\nabla\mathscr{E}(u)}{\|\nabla\mathscr{E}(u)\|}$ and*

$$\mathscr{E}(P(u_s)) - \mathscr{E}(u) < -\frac{s}{2}\|\nabla\mathscr{E}(u)\|.$$

Proof. First, there exists $\varepsilon_0 > 0$ such that zero does not belong to $B(u_0, \varepsilon_0)$ and, for any $u \in B(u_0, \varepsilon_0)$, $\nabla\mathscr{E}(u) \neq 0$. In fact, we can assume there exists $c > 0$ such that, for any $u \in B(u_0, \varepsilon_0)$, $\|u\| \geq c$ and $\|\nabla\mathscr{E}(u)\| \geq c$.

Let $0 < \varepsilon_1 < \frac{1}{2}\varepsilon_0$, for any $u \in B(u_0, \varepsilon_1)$ and $v \in B(u, \varepsilon_1)$, we have $v \in B(u_0, \varepsilon_0)$. So, for any $u \in B(u_0, \varepsilon_1)$ and $v \in B(u, \varepsilon_1)$, zero does not belong to $B(u, \varepsilon_1)$, $\|v\| \geq c$ and $\|\nabla\mathscr{E}(v)\| \geq c$.

Second, let us consider $d_u := -\frac{\nabla\mathscr{E}(u)}{\|\nabla\mathscr{E}(u)\|}$. There exists $\alpha > 0$ such that

$$\left\langle \nabla\mathscr{E}(u_0) | d_{u_0} \right\rangle < -\frac{\|\nabla\mathscr{E}(u_0)\|}{\sqrt{2}} - \alpha.$$

By continuity, possibly taking a smaller ε_1, if $\varepsilon_2 < \frac{1}{2}\varepsilon_1$ then, for any $u \in B(u_0, \varepsilon_2)$ and $v \in B(u, \varepsilon_2)$,

$$\langle \nabla\mathscr{E}(v) | d_u \rangle \leq -\frac{\|\nabla\mathscr{E}(u)\|}{\sqrt{2}}.$$

Let $\varepsilon_3 < \frac{1}{4}\varepsilon_2$, there exist $2^{-1/2} < t_1 < 1 < t_2$ such that, for any $t \in (t_1, t_2)$, $s \in (0, \varepsilon_3)$, $u \in B(u_0, \varepsilon_3)$ and $v \in B(0, \varepsilon_3)$, the function $tu + v + sd_u$ belongs to $B(u, \varepsilon_2)$.

By definition of P, for any $u \in B(u_0, \varepsilon_3) \cap \operatorname{Im} P$, u maximizes $\mathscr{E}$ on

$$(t_1 u, t_2 u) \oplus B_E(0, \varepsilon_3).$$

So, for any $t \in (t_1, t_2)$, $s \in (0, \varepsilon_3)$, $v \in B_E(0, \varepsilon_3)$ and $u \in B(u_0, \varepsilon_3) \cap \operatorname{Im} P$, we have, using the mean value theorem, that there exists $\sigma \in (0, s) \subseteq (0, \varepsilon_3)$ such that

$$\begin{aligned}\mathscr{E}(tu + v + sd_u) - \mathscr{E}(u) &\leq \mathscr{E}(tu + v + sd_u) - \mathscr{E}(tu + v) \\ &= \langle \nabla \mathscr{E}(tu + v + \sigma d_u) | sd_u \rangle < -\frac{s}{\sqrt{2}} \|\nabla \mathscr{E}(u)\|.\end{aligned}$$

As $tu_s = tu + tsd$, we have, for any $t \in (t_1, t_2)$, $0 < s < \varepsilon_3 / t$, $v \in B_E(0, \varepsilon_3)$ and $u \in B(u_0, \varepsilon_3)$ that

$$\mathscr{E}(tu_s + v) - \mathscr{E}(u) < -ts \frac{\|\nabla \mathscr{E}(u)\|}{\sqrt{2}} < -\frac{s}{2} \|\nabla \mathscr{E}(u)\|.$$

It remains to show that there exist $s_0 < \frac{1}{t_2} \varepsilon_3$ and $r_0 < \varepsilon_3$ such that, for any $0 < s < s_0$ and $u \in B(u_0, r_0)$,

$$P(u_s) \in (t_1 u_s, t_2 u_s) \oplus B_E(0, \varepsilon_3) =: A.$$

If not, there would exist $s_n \to 0$ and $u_n \to u_0$ such that $P(u_n + s_n d_{u_n}) = t_n(u_n + s_n d_{u_n}) + h_n$ does not belong to A, where $t_n > 0$ and $h_n \in E$. On one hand, we obtain that $P_{E^\perp}(P(u_n + s_n d_{u_n})) = t_n P_{E^\perp}(u_n + s_n d_{u_n})$ converges to $P_{E^\perp} u_0$. So, for large n, $t_n \in (t_1, t_2)$. On the other hand, $P_E(P(u_n + s_n d_{u_n})) = t_n P_E(u_n + s_n d_{u_n}) + h_n$ converges to $P_E(u_0)$. So, for large n, $h_n \in B(0, r_3)$, which is a contradiction.

So, there exist $0 < s_0 < \frac{1}{t_2} \varepsilon_3$ and $0 < r_0 < \varepsilon_3$ such that, for any $0 < s < s_0$ and $u \in B(u_0, r_0)$, $P(u_s) \in (t_1 u_s, t_2 u_s) \oplus B_E(0, \varepsilon_3)$ and

$$\mathscr{E}(P(u_s)) - \mathscr{E}(u) < -\frac{s}{2} \|\nabla \mathscr{E}(u)\|,$$

which concludes the proof. □

Remark 6.1.4. The value $\frac{1}{2}$ given in Lemma 6.1.3 is not optimal. It can be changed by any number between 0 and 1 (excluded).

6.1.1 Convergence up to a subsequence

With the deformation Lemma 6.1.3, using the generalized MPA, it is possible to construct a stepsize s_n such that the energy $\mathscr{E}$ is decreasing along the obtained sequence $(u_n)_{n\in\mathbb{N}}$. Of course, without loss of generality, we can assume that $\nabla\mathscr{E}(u_n) \neq 0$ for any $n \in \mathbb{N}$ (otherwise we directly get a critical point).

Proposition 6.1.5. *If s_n verifies the inequality about the energy functional given in Lemma 6.1.3 for any $n \in \mathbb{N}$, then the functional $\mathscr{E}$ decreases along the sequence $(u_n)_{n\in\mathbb{N}}$.*

Proof. As $\nabla\mathscr{E}(u_n) \neq 0$, $s_n > 0$ by Lemma 6.1.3. By construction, we have

$$\mathscr{E}(u_{n+1}) - \mathscr{E}(u_n) = \mathscr{E}\left(P\big(u_n - s_n \frac{\nabla\mathscr{E}(u_n)}{\|\nabla\mathscr{E}(u_n)\|}\big)\right) - \mathscr{E}(u_n) < -\frac{s_n}{2}\|\nabla\mathscr{E}(u_n)\| < 0.$$

So, $\mathscr{E}(u_{n+1}) < \mathscr{E}(u_n)$. □

Nevertheless, this is not enough to obtain the convergence to a critical point: we must avoid s_n being arbitrary close to zero without being "mandated" by the functional. To have it, let $u_0 \in \mathscr{H} \setminus E$ and $u_s = u_0 - s\frac{\nabla\mathscr{E}(u_0)}{\|\nabla\mathscr{E}(u_0)\|}$, we set

$$\begin{aligned} S^*(u_0) := \big\{s_0 > 0 : \forall\, 0 < s \leq s_0,\ u_s \neq 0 \text{ and} \\ \mathscr{E}(P(u_s)) - \mathscr{E}(u_0) < -\frac{s}{2}\|\nabla\mathscr{E}(u_0)\|\big\} \end{aligned}$$

and $S(u_0) := S^*(u_0) \cap \big(\frac{1}{2}\sup S^*(u_0), +\infty\big)$.

The set $S^*(u_0)$ is always defined and not-empty as soon as u_0 is not a critical point of $\mathscr{E}$ (thanks to the deformation lemma). Concerning $S(u_0)$, it is not-empty once $\mathscr{E}$ is bounded from below on $\operatorname{Im} P$.

We consider the following generalized mountain pass algorithm:

Algorithm 6.1.6. *1. Choose $u_0 \in \operatorname{Im} P$ and $n \leftarrow 0$;*

2. If $\nabla\mathscr{E}(u_n) = 0$, then stop;
else, $n \leftarrow n+1$ and compute

$$u_{n+1} = P\left(u_n - s_n \frac{\nabla\mathscr{E}(u_n)}{\|\nabla\mathscr{E}(u_n)\|}\right),$$

***where** $s_n \in S(u_n)$;*

3. *Go to step* 2.

The following result gives a "stablility" and uniformity of the algorithm.

Lemma 6.1.7. *If $u_0 \in \operatorname{Im} P$, $\nabla\mathscr{E}(u_0) \neq 0$ and P is continuous at u_0, then there exists an open neighbourhood V of u_0 and $s^* > 0$ such that $S(u) \subseteq [s^*, +\infty)$, for any $u \in V \cap \operatorname{Im} P$.*

Proof. By the uniform deformation Lemma 6.1.3, there exists $s_0 > 0$ and $r_0 > 0$ such that, for any $0 < s \leq s_0$ and $u \in B(u_0, r_0)$, we have $u - s\frac{\nabla\mathscr{E}(u)}{\|\nabla\mathscr{E}(u)\|} \neq 0$, $\nabla\mathscr{E}(u) \neq 0$ and

$$\mathscr{E}\left(P\big(u - s\frac{\nabla\mathscr{E}(u)}{\|\nabla\mathscr{E}(u)\|}\big)\right) - \mathscr{E}(u) < -\frac{s}{2}\|\nabla\mathscr{E}(u)\|.$$

So, for any $u \in B(u_0, r_0)$, $s_0 \in S^*(u)$. It follows that $S(u) \subseteq (\frac{s_0}{2}, +\infty)$. It suffices to take $s^* < s_0/2$. $\square$

The result 6.1.9 is a key to obtain the proof of the convergence up to a subsequence. Before proving that result, let us establish the following lemma.

Lemma 6.1.8. *Let $c > 0$. For any $u, v \in \mathscr{H}$ such that $\|u\| \geq c$ and $\|v\| \geq c$,*

$$\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leq c^{-1}\|u - v\|.$$

Proof. This is an immediate consequence of the following simple calculation:

$$\begin{aligned}
\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\|^2 &= 2 - 2\frac{\langle u|v\rangle}{\|u\|\|v\|} \\
&= \frac{1}{\|u\|\|v\|}(2\|u\|\|v\| - 2\langle u|v\rangle) \\
&\leq \frac{1}{\|u\|\|v\|}(\|u\|^2 + \|v\|^2 - 2\langle u|v\rangle) \\
&= \frac{\|u - v\|^2}{\|u\|\|v\|}.
\end{aligned}$$

$\square$

Lemma 6.1.9. *Let $(u_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ given by the generalized MPA 6.1.6. Assume that $\inf_{u\in\operatorname{Im} P}\|P_{E^\perp}u\| > 0$ and that P is continuous.*
If $\sum_{n=0}^{+\infty} s_n < +\infty$ then $(u_n)_{n\in\mathbb{N}}$ converges in $\mathscr{H}$.

Proof. To simplify the notations, we denote $u^* := P_{E^\perp} u$. Let $v_n := \frac{u_n^*}{\|u_n^*\|}$, $a := \inf_{u \in \operatorname{Im} P} \|u^*\| > 0$ and $d_n := -\frac{\nabla \mathscr{E}(u_n)}{\|\nabla \mathscr{E}(u_n)\|}$.

First, as $\|u_n^*\| \geq a > 0$ and there exists n_0 such that, for any $n \geq n_0$, $s_n < \frac{a}{2}$, we have

$$\|u_n^* + s_n d_n^*\| \geq \|u_n^*\| - s_n \|d_n^*\| \geq \|u_n^*\| - s_n \geq \frac{a}{2} =: b > 0.$$

Second, for any $n \in \mathbb{N}$, there exists $t_n > 0$ and $h_n \in E$ such that

$$\begin{aligned} v_{n+1} &= \frac{(P(u_n + s_n d_n))^*}{\|(P(u_n + s_n d_n))^*\|} = \frac{(t_n u_n + t_n s_n d_n + h_n)^*}{\|(t_n u_n + t_n s_n d_n + h_n)^*\|} \\ &= \frac{t_n u_n^* + t_n s_n d_n^*}{\|t_n u_n^* + t_n s_n d_n^*\|} = \frac{u_n^* + s_n d_n^*}{\|u_n^* + s_n d_n^*\|}. \end{aligned}$$

So, for any $n > n_0$, by Lemma 6.1.8, we have

$$\|v_{n+1} - v_n\| \leq 2b^{-1} \|s_n d_n^*\| \leq 2b^{-1} s_n.$$

So, $(v_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. As $P(v_n) = P(u_n^*) = u_n$, by continuity of P, $(u_n)_{n \in \mathbb{N}}$ converges. □

Remark 6.1.10. In the previous proof of Lemma 6.1.9, if we wanted to work with the projection on the nodal Nehari set $P_{\mathscr{M}}$ defined in the Introduction of this chapter, we could not remove the projection in the computation of v_{n+1}. This explains why the proof of convergence is not known at the moment in this case.

Theorem 6.1.11. *If zero does not belong to $\overline{\operatorname{Im} P}$, P is continuous, $\mathscr{E}$ satisfies the Palais-Smale condition in $\operatorname{Im} P$ and $\inf_{u \in \operatorname{Im} P} \mathscr{E}(u) > -\infty$, then $(u_n)_{n \in \mathbb{N}}$ given by the generalized mountain pass algorithm 6.1.6 possesses a subsequence converging to a critical point of $\mathscr{E}$. Moreover, all accumulation points of $(u_n)_{n \in \mathbb{N}}$ are critical points of $\mathscr{E}$.*

Proof. Let us start by showing that $(\nabla \mathscr{E}(u_n))_{n \in \mathbb{N}}$ converges to zero up to a subsequence. If not, we could assume there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, $\|\nabla \mathscr{E}(u_n)\| > \delta$. Then, for any $n \geq n_0$, we have

$$\mathscr{E}(u_{n+1}) - \mathscr{E}(u_n) \leq -\frac{1}{2} s_n \delta.$$

Thus, by adding up,

$$\lim_{n\to\mathbb{N}} \mathscr{E}(u_n) - \mathscr{E}(u_{n_0}) = \sum_{n=n_0}^{+\infty} \mathscr{E}(u_{n+1}) - \mathscr{E}(u_n) \leq -\frac{\delta}{2} \sum_{n=n_0}^{+\infty} s_n.$$

On one hand, as the left-hand side is a real number ($\mathscr{E}$ is bounded from below on $\operatorname{Im} P$ and decreasing on $(u_n)_{n\in\mathbb{N}}$), we have $\sum_{n=0}^{+\infty} s_n < +\infty$. So, by Lemma 6.1.9, $u_n \to u^* \in \overline{\operatorname{Im} P}$ and $\|\nabla\mathscr{E}(u^*)\| \geq \delta$. On the other hand, by continuity of P, we have $P(u^*) = u^*$ and, so, $u^* \in \operatorname{Im} P$. By Lemma 6.1.7, there exists a neighbourhood V of u^* and $s^* > 0$ such that $S(u) \subseteq [s^*, +\infty)$, for any $u \in V$. So, there exists n_0 such that, for any $n \geq n_0$, $s_n \geq s^*$ whence $\sum_{n=0}^{+\infty} s_n$ does not converge, which is a contradiction.

Therefore, there exists a sequence $(u_{n_k})_{k\in\mathbb{N}} \subseteq (u_n)_{n\in\mathbb{N}}$ such that $\|\nabla\mathscr{E}(u_{n_k})\|$ converges to 0 when $k \to +\infty$. As $\mathscr{E}$ satisfies the Palais-Smale condition, $(u_{n_k})_{k\in\mathbb{N}}$ possesses a subsequence converging to a critical point of $\mathscr{E}$.

Concerning the second statement of the thesis, the argument is very similar. Let $(u_{n_k})_{k\in\mathbb{N}}$ be a convergent subsequence and assume on the contrary that $u = \lim_{k\to\infty} u_{n_k}$ is not a critical point of $\mathscr{E}$. In that case, on one hand, there exists $\delta > 0$ and $k_1 \in \mathbb{N}$ such that, for any $k \geq k_1$, $\|\nabla\mathscr{E}(u_{n_k})\| \geq \delta$. By Lemma 6.1.3, we have

$$\forall k \geq k_1, \quad \mathscr{E}(u_{n_{k+1}}) - \mathscr{E}(u_{n_k}) \leq -\frac{1}{2}\delta s_{n_k}.$$

On the other hand, as $u \in \operatorname{Im} P$, we have by Lemma 6.1.7 that

$$\exists\, s^* > 0,\ \exists k_2 \in \mathbb{N},\ \forall k \geq k_2,\ s_n \geq s^*.$$

So, for large k, $\mathscr{E}(u_{n_{k+1}}) - \mathscr{E}(u_{n_k}) \leq -\frac{1}{4}\delta s^*$, which is a contradiction because $(\mathscr{E}(u_n))_{n\in\mathbb{N}}$ is a convergent sequence. □

6.1.2 Convergence

In this section, under a "localization" assumption, we prove that the entire sequence $(u_n)_{n\in\mathbb{N}}$ given by the generalized mountain pass algorithm 6.1.6 converges to a non-zero solution of Problem (*SP*). It explains why we changed the definition of the stepsize s_n given in the paper [66].

Assume that we are able to locate a non-zero solution u of Problem (*SP*) with the following property: u is the unique critical point of $\mathscr{E}$ in the ball $B(u,\delta)$, for some $\delta > 0$. Let us denote $S(u,r) := \{v : \ \|v-u\| = r\}$.

Theorem 6.1.12. *If there exists $n^* \in \mathbb{N}$ such that $u_{n^*} \in B(u,\delta)$ and $\mathscr{E}(u_{n^*}) < a := \inf_{v\in S(u,\delta)\cap \operatorname{Im} P} \mathscr{E}(v)$ then the sequence $(u_n)_{n\in\mathbb{N}}$ converges to u.*

Proof. For any $m > n^*$, $u_m \in B(u,\delta)$. If not, as $u_{n^*} \in B(u,\delta)$, there exists $m \geq n^*$ such that $u_m \in B(u,\delta)$ and $u_{m+1} = P\big(u_m - s_m \frac{\nabla\mathscr{E}(u_m)}{\|\nabla\mathscr{E}(u_m)\|}\big) \notin B(u,\delta)$, with $s_m \in S(u_m)$.

So, by continuity, there exists $0 < s \leq s_m$ such that $P\big(u_m - s\frac{\nabla\mathscr{E}(u_m)}{\|\nabla\mathscr{E}(u_m)\|}\big) \in S(u,\delta)\cap \operatorname{Im} P$. This is a contradiction because, by the definition of s_m and as $\mathscr{E}$ is decreasing on $(u_n)_{n\in\mathbb{N}}$, we have $a \leq \mathscr{E}\big(P(u_m - s\frac{\nabla\mathscr{E}(u_m)}{\|\nabla\mathscr{E}(u_m)\|})\big) \leq \mathscr{E}(u_m) \leq \mathscr{E}(u_{n^*}) < a$.

As u is the unique critical point in $B(u,\delta)$, by Theorem 6.1.11, u is the unique accumulation point of $(u_n)_{n\in\mathbb{N}}$. So, u_n converges to u.

□

6.2 Application and conjecture

Let us come back to the problem

$$\begin{cases} -\Delta u(x) + V(x)u(x) = f(u(x)), & x\in\Omega,\\ u(x) = 0, & x\in\partial\Omega, \end{cases}$$

where V satisfies assumption (V_1) and f respects assumptions $(F_1)-(F_4)$.

Solutions are the critical points of the energy functional

$$\mathscr{E} : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega \big(|\nabla u(x)|^2 + V(x)u(x)^2\big)\,\mathrm{d}x - \int_\Omega F(u(x))\,\mathrm{d}x, \quad (6.3)$$

where $F(u) := \int_0^u f(s)\,\mathrm{d}s$.

Let us denote the decomposition $H_0^1(\Omega) = H^{(-)} \oplus H^{(+)}$ corresponding to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum.

We work with the norm induced by the following inner product

$$\langle u|v\rangle = \int_\Omega \nabla u^{(+)}\nabla v^{(+)} + V(x)u^{(+)}v^{(+)} - \int_\Omega \nabla u^{(-)}\nabla v^{(-)} + V(x)u^{(-)}v^{(-)}.$$

In the aim to implement the generalized mountain pass algorithm 6.1.6, the space $\mathscr{H}$ is $H_0^1(\Omega)$, the subspace E is $H^{(-)}$ and the function P denotes the continuous projection from $H(\Omega) \setminus H^{(-)}$ to the generalized Nehari manifold $\mathscr{N}_G$ defined by

$$P : H_0^1(\Omega) \setminus H^{(-)} \to \mathscr{N}_G : u \mapsto P(u)$$

such that $P(u)$ maximizes $\mathscr{E}$ on $H^*(u) = \mathbb{R}^+ u \oplus H^{(-)} = \mathbb{R}^+ u^{(+)} \oplus H^{(-)}$. This projection P will be computed by a standard maximization procedure.

6.2.1 Convergence of the generalized MPA for Schrödinger problems

To obtain results of Section 6.1 on the convergence of the mountain pass algorithm 6.1.6, we need to verify the following assumptions on $\mathscr{E}$:

1. it is standard to show that $\mathscr{E} \in \mathscr{C}^1(\Omega)$;
2. $\mathscr{E}$ verifies the Palais-Smale condition on $\operatorname{Im} P$ (see [64]);
3. $\inf_{u \in \operatorname{Im} P} \mathscr{E}(u) > -\infty$: actually $\mathscr{E}$ is bounded from below by 0 on $\operatorname{Im} P$, see [64];
4. 0 does not belong to $\overline{\operatorname{Im} P}$: it comes from the fact that 0 is a strict local minimum on $E^\perp = H^{(+)}$ (see e.g. [64]);
5. $\inf_{u \in \operatorname{Im} P} \|u^{(+)}\| > 0$: otherwise, there would exist a sequence $(u_n)_{n \in \mathbb{N}} \subseteq \operatorname{Im} P$ such that $\|u_n^{(+)}\| \to 0$. As $\mathscr{E}(u_n) > 0$, we have
$$\|u_n^{(+)}\|^2 \geq \|u_n^{(-)}\|^2 + \frac{1}{p} \int_\Omega |u_n|^p \,\mathrm{d}x \geq \|u_n^{(-)}\|^2.$$
So, we have $u_n^{(-)} \to 0$ and $u_n \to 0$. This is impossible because $\operatorname{Im} P$ stays away from zero.

So, we obtain the convergence of the generalized mountain pass algorithm 6.1.6 for this current problem when the domain Ω is bounded.

Let us now sketch what happens about the convergence when $\Omega = \mathbb{R}^N$. In that case, we assume that hypothesis (PA) is verified.

Proposition 6.2.1. *Let $(u_n)_{n \in \mathbb{N}}$ given by the mountain pass algorithm 6.1.6. There exists a sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}^N$ such that $(u_n(\cdot + x_n))_{n \in \mathbb{N}}$ weakly converges up to a subsequence to a nontrivial critical point u_* of $\mathscr{E}$. Moreover, if $\mathscr{E}(u_n) \to \inf_{u \in \operatorname{Im} P} \mathscr{E}(u)$, then the above convergence is strong.*

Proof. The only trouble in repeating the previous argument is that the Palais-Smale condition does not hold anymore. Nevertheless, as $(\mathscr{E}(u_n))_{n\in\mathbb{N}}$ is decreasing and bounded from below by zero, we have that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$ (see [64]). Thus, we obtain that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^N$ sucht that $u(\cdot + x_n)$ weakly converges, up to a subsequence, to u. Intuitively, the translations "bring back" some mass that u_n may loose at infinity.

Without loss of generality, we can assume that $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}^N$. As f and V are 1-periodic by assumption (PA), we have that $\mathscr{E}(u_n(\cdot + x_n)) = \mathscr{E}(u_n)$ and $\nabla\mathscr{E}(u_n(\cdot + x_n)) = \nabla\mathscr{E}(u_n)$. Thus, we can conclude that u^* is a critical point of $\mathscr{E}$.

Finally, if $\mathscr{E}(u_n) \to \inf_{u\in \mathrm{Im}\, P} \mathscr{E}(u)$, no mass can be lost at infinity (if not, this mass will increase the energy) and so we obtain our convergence.

□

6.2.2 Illustration and conjecture

For illustration, let us consider $V(x) = -\lambda \in \mathbb{R}$ where λ is not an eigenvalue of $-\Delta$.

Let us remark that $H^{(-)}$ is formed by eigenfunctions of $-\Delta$ with negative eigenvalues. Let us illustrate this on the the square $(0,1)^2$. In practice, we need to compute with great accuracy the projection P to obtain satisfying results. As the generalized MPA 6.1.6 requires lots of projections on $H^{(+)}$ and $H^{(-)}$, the algorithm can be very slow. A satisfying approximation could only be found in a suitable time only for the case where $\lambda_1 < \lambda < \lambda_2$ and for a low number of nodes in the used for the finite element method. Nevertheless, we believe the results are convincing enough to make some conjecture about expected symmetries of ground state solutions.

Figure 6.1 depicts a non-zero solution u of the problem $-\Delta u - \lambda u = u^3$ with Dirichlet boundary conditions, for $\lambda_1 < \lambda < \lambda_2$. To be precise, λ equals 21. We can observe that the solution seems to be odd with respect to a diagonal and even with respect to the orthogonal one. So, it respects the symmetries of the least energy nodal solutions in the case where $\lambda = 0$ (see Figure 1.6).

In Table 6.1, we present the characteristics of the solution.

In Chapter 1, it is also proved in Theorem 1.6.2 that, for problem $-\Delta u = |u|^{p-2}u$ with DBC, if p is close to 2, ground state solutions must respect sym-

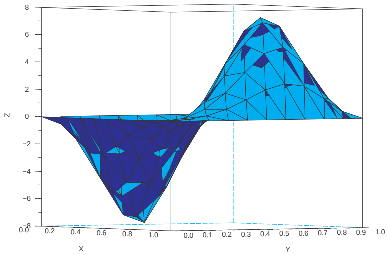


Figure 6.1: MPA solution for an indefinite problem on a square

	Initial function	$\min u$	$\max u$	$\mathscr{E}(u^+)$	$\mathscr{E}(u^-)$
S	$xy(x-1)(y-1)$	-7.33	7.36	33.8	33.8

Table 6.1: Characteristics of the ground state solution for a indefinite problem

metries of its projection on E_1 (i.e. first eigenspace of $-\Delta$). Also, by Theorem 1.4.5, least energy nodal solutions must respect the symmetries of its projection on E_2 (the second one). So, by this study and previous computations, we make the following conjecture.

Conjecture 6.2.2. If $\lambda_{n-1} < \lambda < \lambda_n$, ground state solutions must respect symmetries of its projection on E_n, the n^{th} eigenspace of $-\Delta$.

Appendix A

Some classical results

In this appendix, we only list some important results used several times during the proofs presented in this work.

The two first results are classical in functional analysis. We mainly used them to apply the Lebesgue's dominated convergence theorem or to prove the convergence of some sequences.

Proposition A.1. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of the Banach space X and $u \in X$. If, from any subsequence $(v_n)_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, there exists a subsequence $(w_n)_{n\in\mathbb{N}}$ of $(v_n)_{n\in\mathbb{N}}$ converging to u, then u_n converges to u.

Proposition A.2. Let $1 \leq p < +\infty$. If $u_n \to u$ in $L^p(\Omega)$ then, up to a subsequence, there exists $f \in L^p(\Omega)$ such that $|u_n| \leq f$ almost everywhere.

The following results are classical in the Sobolev space theory. To start, we give a characterization of λ_n, the eigenvalues of $-\Delta$. This result is called the Poincaré's principle. Let us fix E_i the eigenspaces related to λ_i.

Theorem A.3. For any $n \in \mathbb{N}$,

$$\begin{aligned}\lambda_n = \min\Big\{ \int_\Omega |\nabla u|^2 : u \in H_0^1(\Omega), \int_\Omega u^2 \,\mathrm{d}x = 1,\\ \int_\Omega u e_1 \,\mathrm{d}x = \cdots = \int_\Omega u e_{n-1} = 0,\\ \text{for any } e_i \in E_i \Big\}.\end{aligned}$$

Also due to H. Poincaré, we can work with "more general" Sobolev spaces.

Theorem A.4. Let Ω be an open bounded domain of $\mathbb{R}^N$ and $1 \leq q < +\infty$. There exists a positive constant C such that, for any[1] $u \in W_0^{1,q}(\Omega)$,

$$\|u\|_q \leq C\|\nabla u\|_q.$$

The two following results give the Sobolev's embeddings.

Theorem A.5. If $1 \leq p < N$, then there exists a positive constant C such that, for any test functions[2] u,

$$\|u\|_{L^{\frac{pN}{N-p}}} \leq C\|\nabla u\|_{L^p}.$$

Theorem A.6.

- Let $\Omega \subseteq \mathbb{R}^N$ be a domain respecting the cone condition[3]. If $1 \leq q < N$ and $q \leq s \leq \frac{qN}{N-q}$ then $W^{1,q}(\Omega) \subseteq L^s(\Omega)$ and the embedding is continuous.
- Let $\Omega \subseteq \mathbb{R}^N$ be a local Lipschitz domain. If $N < q < +\infty$ then $W^{1,q} \subseteq \mathscr{C}(\overline{\Omega})$ and the embedding is continuous.

The last one gives to us the Rellich's embedding.

Theorem A.7.

- Let $\Omega \subseteq \mathbb{R}^N$ be a domain respecting the cone condition. If $1 \leq q < N$ and $1 \leq s < q^* = \frac{qN}{N-q}$ then $W^{1,q}(\Omega) \subseteq L^s(\Omega)$ and the embedding is compact.
- Let $\Omega \subseteq \mathbb{R}^N$ be a local Lipschitz domain. If $N < q < +\infty$ then $W^{1,q} \subseteq \mathscr{C}(\overline{\Omega})$ and the embedding is compact.

Let us remark that if we are working with spaces $W_0^{1,q}(\Omega)$, the conclusions of theorems A.6 and A.7 are valid for any domains Ω. Readers can find a proof in [2].

---

[1] Closure in $L^q(\Omega)$ of the space $\mathscr{C}_0^2(\Omega)$ for the classical norm $(\int_\Omega |\nabla u|^q)^{1/q}$.

[2] Functions in $\mathscr{C}^{+\infty}$ with compact support.

[3] There exists a finite cone K such that each point $x \in \Omega$ is the vertex of a finite cone K_x contained in Ω and congruent to K.

Notations

Δ	Laplacian operator
∇	Gradient operator
div	Divergence operator
$H(\Omega)$	Sobolev space $H_0^1(\Omega)$(resp. $H^1(\Omega)$)
	depending on Problem with DBC (resp. NBC)
H	$H(\Omega)$ in short
$\|u\|^2$	$\int_\Omega \|\nabla u\|^2$ in $H_0^1(\Omega)$ or $\int_\Omega \|\nabla u\|^2 + u^2$ in $H^1(\Omega)$
λ_i	i^{th} eigenvalue in H of $-\Delta$ with DBC or $-\Delta + \mathrm{id}$ with NBC
E_i	Eigenspace related to λ_i
e_i	Eigenfunction in E_i with $\|e_i\| = 1$
$\|u\|_p$	Traditional norm in L^p: $\left(\int_\Omega \|u\|^p\right)^{1/p}$
$\|.\|_X$	Traditional norm in X
S	Unit sphere of H
S^N	Unit sphere of $\mathbb{R}^N$
$S(a,r)$	Sphere centered in a with radius r in H
B	Unit ball of H
B^N	Unit ball of $\mathbb{R}^N$
$B(a,r)$	Open ball centered in a with radius r in H

$B[a,r]$	Closed ball centered in a with radius r in H
$\langle u\|v\rangle$	Inner product in H: $\int_\Omega \nabla u \nabla v$ if $H_0^1(\Omega)$ and $\int_\Omega \nabla u \nabla v + uv$ if $H^1(\Omega)$, unless said otherwise
$f'(.)$	Derivative of $f : \mathbb{R} \to \mathbb{R}$
$\mathrm{d}\mathscr{E}(.)$	Frechet derivative of $\mathscr{E}$
$\langle \mathrm{d}\mathscr{E}(u), v\rangle$	Derivative of $\mathscr{E}$ at u taken in the direction v
$\langle \mathrm{d}^2\mathscr{E}(u), v, w\rangle$	Second derivative of $\mathscr{E}$ at u taken in the directions v and w
$\partial_u \mathscr{E}$	Partial derivative of $\mathscr{E}$ related to u
$X^\perp$	Orthogonal space of X in H
P_X	Orthogonal projection from H to X
$d(A,B)$	Distance between A and B
∂A	Boundary of A
X'	Dual space of X (continuous and linear functionals)
u^+	$\max(0,u)$
u^-	$\min(0,u)$
$\Subset$	Compact embedding
$\overline{A}$	Closure of set A
∂_ν	Derivative in the outer normal direction
$M(\mathbb{R})$	Set of Borel-measurable functions from $\mathbb{R}$ to $\mathbb{R}$
$\mathscr{C}(X)$	Set of continuous functions from X to $\mathbb{R}$
$\mathscr{C}^n(X)$	Set of fns n times continuous differentiable from X to $\mathbb{R}$
$\mathscr{C}^n(X,Y)$	Set of fns n times continuous differentiable from X to Y
$\mathscr{C}_0^n(X)$	Set of fns n times differentiable with compact support
$\mathscr{C}^{1,\alpha}$	$\mathscr{C}^1$-functions with α-Hölder continuous derivative
$\mathscr{C}^{1,\alpha}_{\mathrm{loc}}$	$\mathscr{C}^1$-functions with α-Hölder continuous derivative on any compact
$\sigma(f)$	Spectrum of operator f

$\mathrm{Im}(f)$	Image of f

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