Nonlinear Schrödinger problems: multiplicity and symmetries of positive solutions

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Joint work with D. Bonheure
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Problem

Let $\Omega \subseteq \mathbb{R}^N$ open bounded, $N \geqslant 2$ and $2 < p < 2^* := \frac{2N}{N-2}$. We consider

$$(P) \begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } \Omega, \\ \mathrm{d}u = 0, & \text{in } \partial\Omega. \end{cases}$$

Solutions are **critical points** of the **energy functional**

Energy functional

- $\mathcal{E}_p : H^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 + u^2 - \frac{1}{p}\int_\Omega |u|^p$
- $\partial\mathcal{E}_p(u) : H^1(\Omega) \to \mathbb{R} : v \mapsto \int_\Omega \nabla u \nabla v + uv - \int_\Omega |u|^{p-2}uv$

$\|u\|^2_{H^1} := \int_\Omega |\nabla u|^2 + u^2$
0 is always solution
Goal: study of (positive) solutions

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Schedule

1. p small?
2. p large?
   - Ground state are not symmetric,
   - bifurcation results,
   - multiplicity results.

Schedule

1. p small?
   1. Positive ground state solutions:
      - same symmetries as projection on E_1 (first eigenspace of $-\Delta + \mathrm{id}$)
      - moreover, it is 1
   2. Non-zero positive solutions are constant
2. p large?

Existence of Ground State for $\begin{cases} -\Delta u + u = |u|^{p-2}u, \ \textit{in } \Omega \\ \mathrm{d}u = 0, \ \textit{on } \partial\Omega. \end{cases}$.

The energy functional is not bounded from below.

Mountain-Pass theorem (Z. Nehari, A. Ambrosetti, P. H. Rabinowitz)

There exists a ground state solution. It is an one-signed function.

Sketch:

- The energy functional $\mathcal{E}_p$ possesses a Mountain-Pass structure
- $\exists u_0 \neq 0, \ \mathcal{E}_p(u_0) = \inf_{u\neq 0} \max_{\lambda>0} \mathcal{E}_p(\lambda u) = \inf_{u\in\mathcal{N}_p} \mathcal{E}_p(u)$
- For any sign-changing solution u: $\mathcal{E}_p(u^{\pm}) < \mathcal{E}_p(u)$, where $u^+ := \max(u, 0)$ and $u^- := \min(u, 0)$

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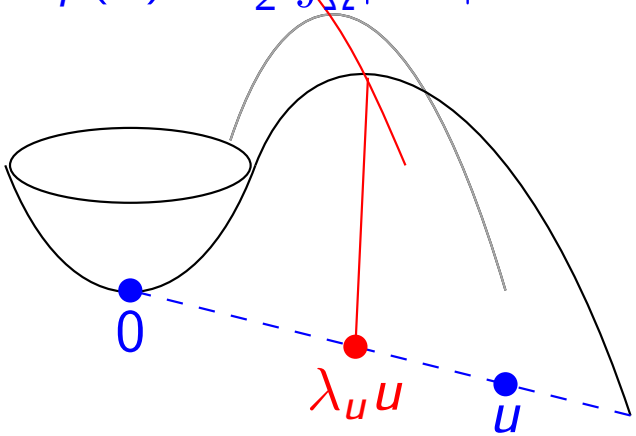

$\mathcal{N}_p := \{u \neq 0 : (u, \mathcal{E}(u)) \in$ "Top of the Mountain" $\}$ and is called

"Nehari manifold"

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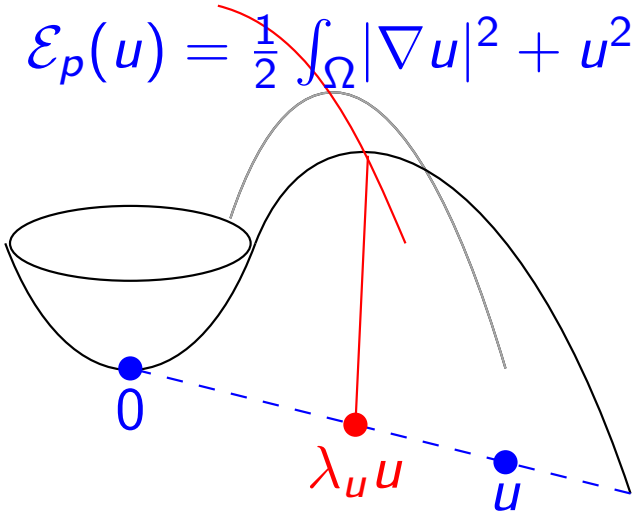

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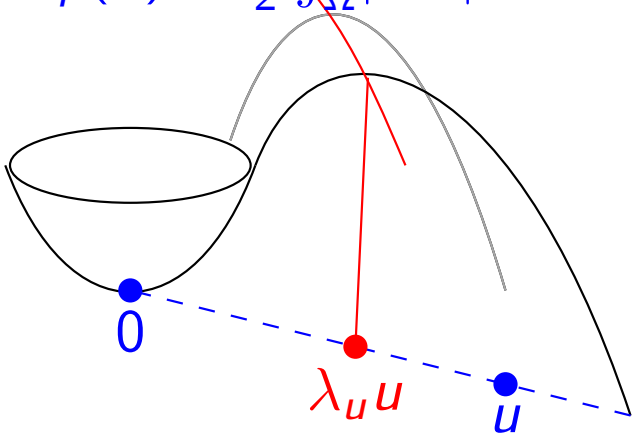

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Remarks on $\mathcal{N}_p$

- $\mathcal{N}_p$ is formed by all the non-zero functions u such that $\partial\mathcal{E}(u)u = 0$...one maximizes energy in the direction of u
- If $u \in \mathcal{N}_p$: $\mathcal{E}_p(u) = (\frac{1}{2} - \frac{1}{p})\|u\|^2$
- $\mathcal{N}_p$ is a manifold of codimension 1. So, Morse index of a ground state solution is 1.

Symmetry result for ground state

Let us remark that the seminal moving planes method does not work.

Theorem (D. Bonheure, V. Bouchez, G., C. Troestler, J. Van Schaftingen)

For p close to 2, *for any reflection R s.t. $R(\Omega) = \Omega$, it* **respects the symmetry or antisymmetry of functions in E_1** *with respect to R.*

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- *If $\lambda_2(\Omega)$ is* **simple**, **for** *$p \approx 2$ and* **l.e.n.s.**,
  - *l.e.n.s.* **respects the symmetry or antisymmetry of functions in** *E_2.*
  - *l.e.n.s. is unique up to a multiplicative factor of value -1*
- *For general domains, it respects symmetries of its projection on E_2*

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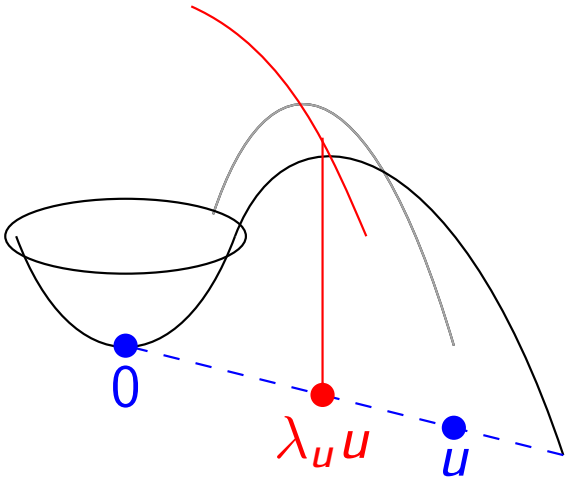
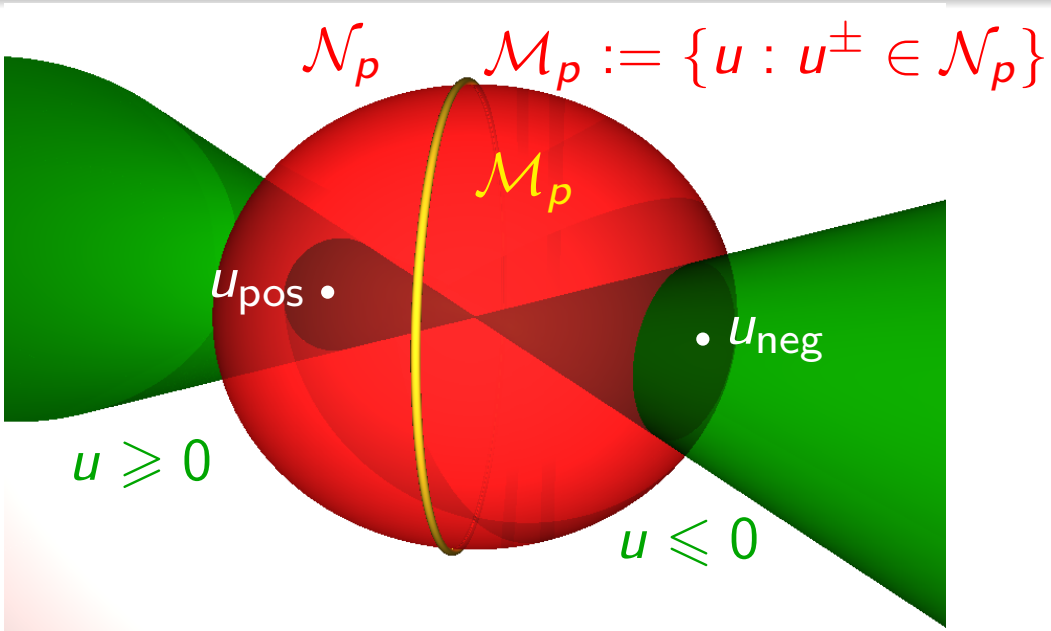
Existence of nodal solution

Theorem (A. Castro, J. Cossio, J. M. Neuberger, '97)

There exists a nodal solution with minimal energy.





Projection: $sH^1 \setminus \{0\} \to \mathcal{M}_p : u \mapsto P_{\mathcal{N}_p}(u^+) + P_{\mathcal{N}_p}(u^-)$.

Remark 1: One maximizes energy in $\{tu^+ + su^- : t, s \geq 0\}$

Sketch of the proof: "Simple" case (1/3)

$$\textbf{(P)} \begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } \Omega \\ \mathrm{d}u = 0, & \text{in } \partial\Omega \end{cases}$$

Up to a rescaling by $\lambda_2^{\frac{-1}{p-2}}$, studying symmetries for the problem (P) is the same that studying symmetries for the problem

$$\textbf{(P2)} \begin{cases} -\Delta u + u = \lambda_2 |u|^{p-2}u, & \text{in } \Omega \\ \mathrm{d}u = 0, & \text{in } \partial\Omega \end{cases}$$

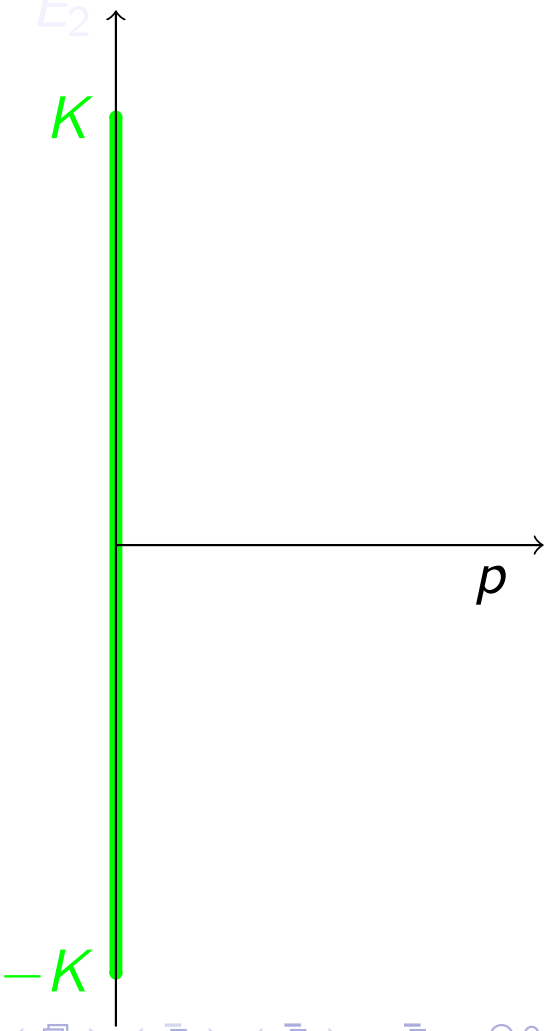
Let $(u_p)_{p>2}$ a **family of least energy nodal solutions** of (P2).

Why (**P2**)

To obtain that $(u_p)_{p>2}$ is **bounded** in H^1 and **away** from 0.

Sketch of the proof (2/2)

1. $\|u_p\| \leqslant K \Rightarrow u_p \rightharpoonup u_0$ up to a subsequence
   $\Rightarrow u_p \to u_0$ up to a subsequence
2. $u_0 \in E_2$
3. u_p stays away from 0
4. computing $u_{p,\lambda} := \frac{u_p}{\|u_p\|}$ s.t.
   $(u_{p,\lambda}, \lambda_2 \|u_p\|^{p-2}) \to (\pm e_2, \lambda_2)$
5. By IFT, the curve starting from e_2 and solution of
   $$\begin{cases} \Delta u(x) + u + \lambda |u(x)|^{p-2} u(x) = 0, \\ \mathrm{d}u(x) = 0, \\ \|u\| = 1 \end{cases}$$
   is unique
6. symmetries of e_2 are respected



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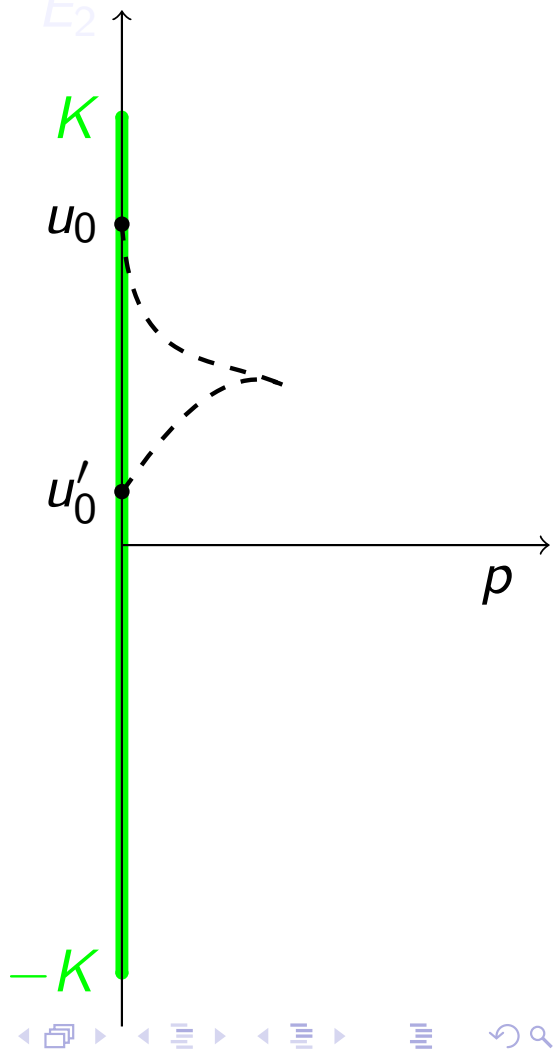

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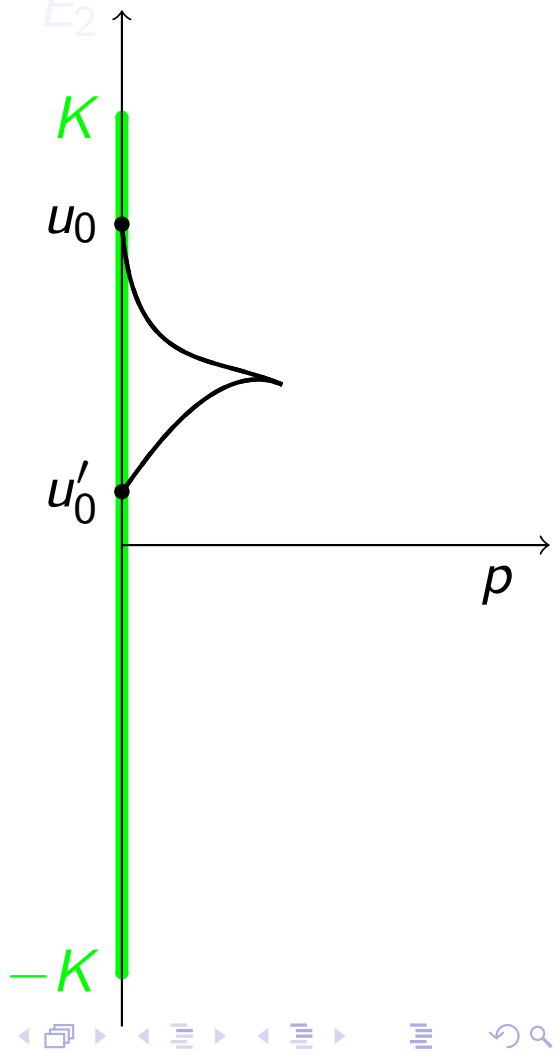

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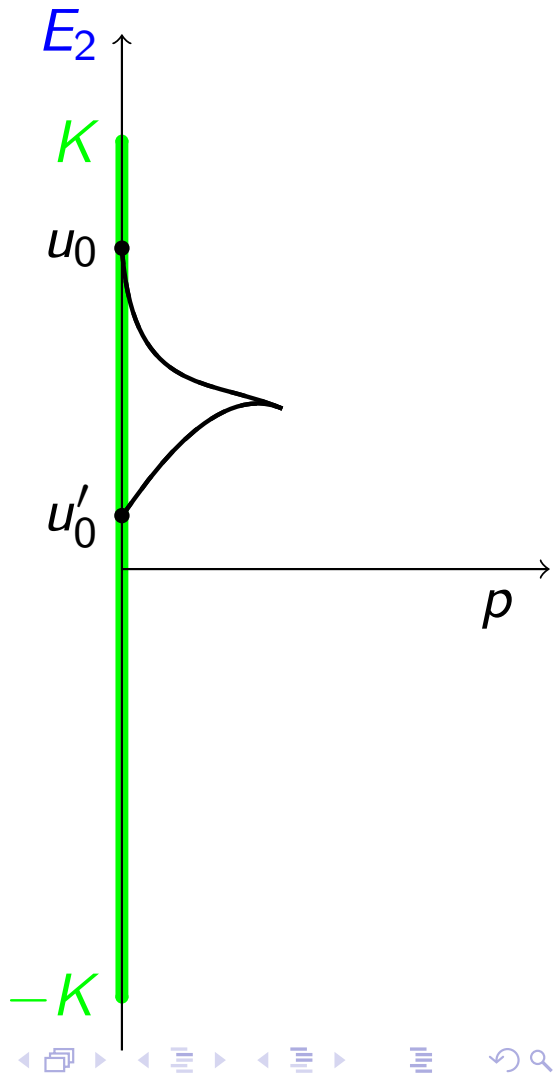

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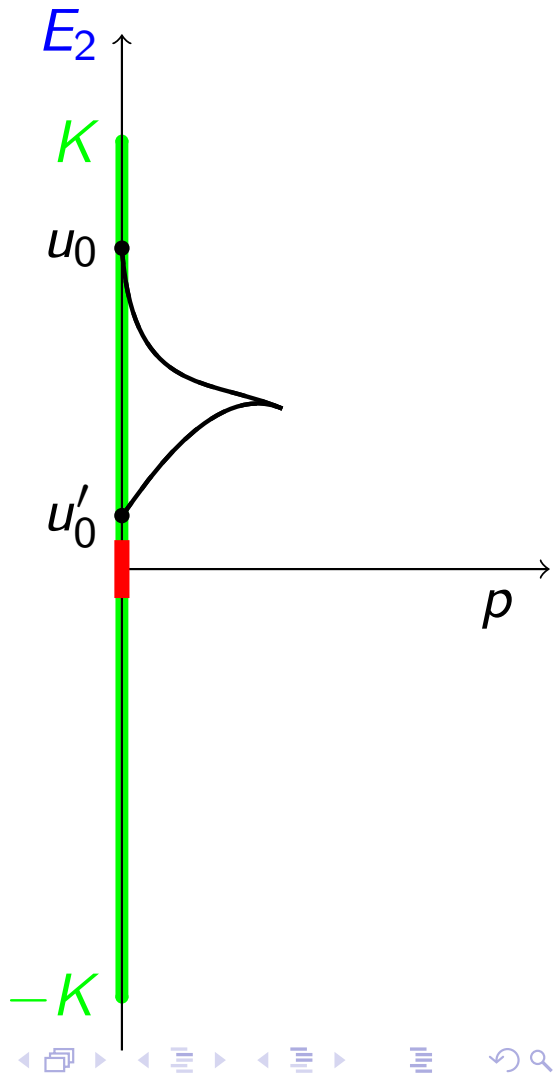

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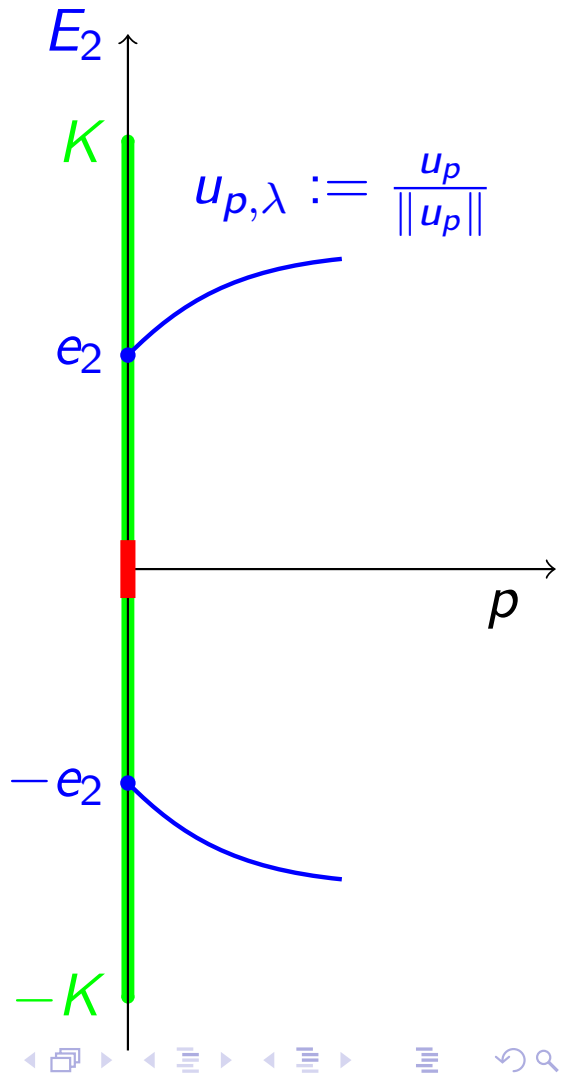

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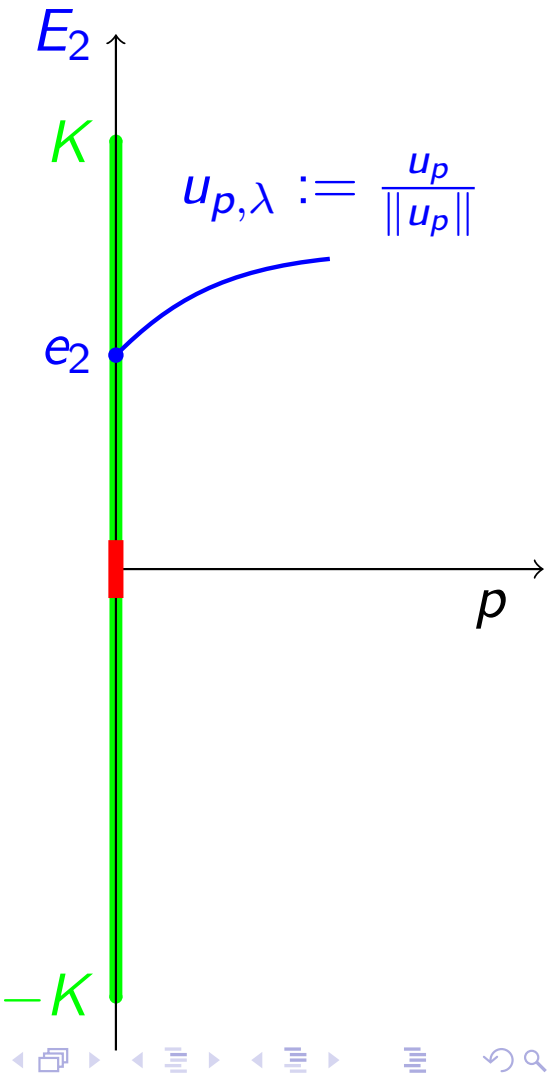

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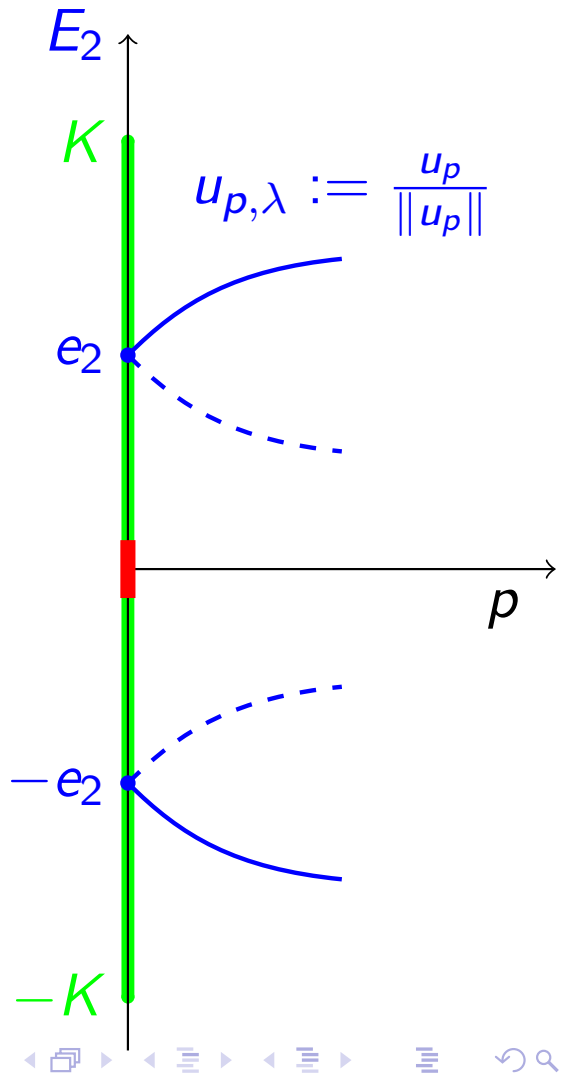

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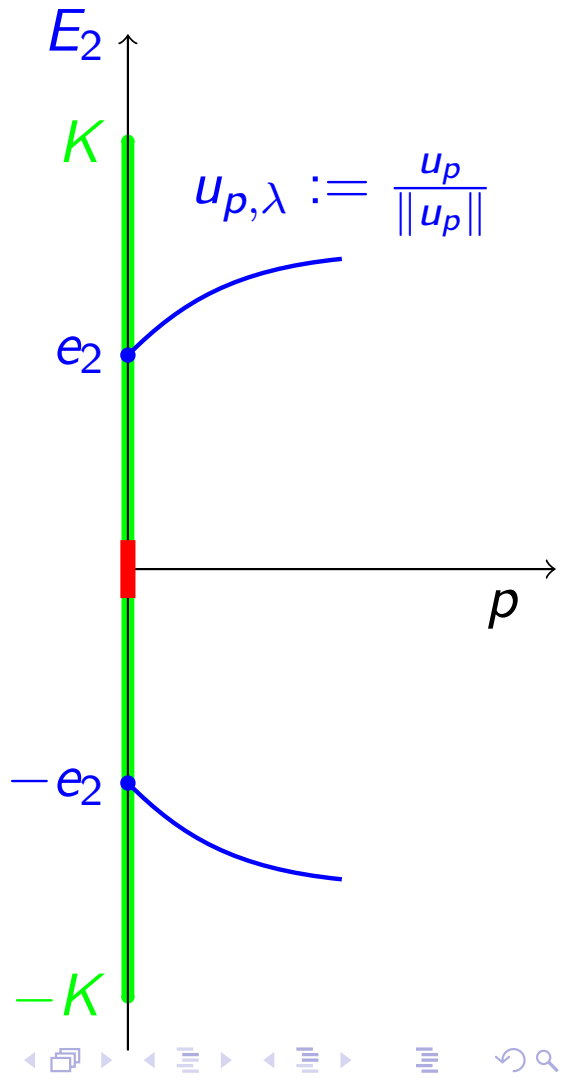


Upper bounded

Lemma

The family $(u_p)_{p>2}$ is bounded.

- Let $e_2 \in E_2 \setminus \{0\}$, we obtain

$$\mathcal{E}_p(u_p) = \left(\frac{1}{2} - \frac{1}{p}\right) \|u_p\|^2 \leq \left(\frac{1}{2} - \frac{1}{p}\right) \{t_p^{+2} \|e_2^+\|^2 + t_p^{-2} \|e_2^-\|^2\}$$

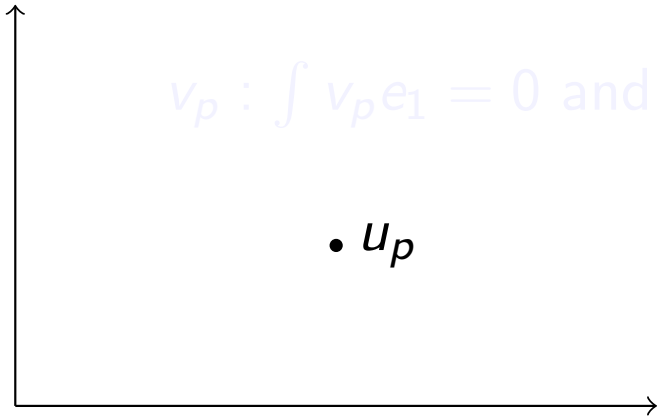
- $t_p^+ = \left(\frac{\int_\Omega |\nabla e_2^+|^2 + (e_2^+)^2}{\lambda_2 \int_\Omega (e_2^+)^{p-1}}\right)^{\frac{1}{p-2}}$ and $t_p^- = \left(\frac{\int_\Omega |\nabla e_2^-|^2 + (e_2^-)^2}{\lambda_2 \int_\Omega (e_2^-)^{p-1}}\right)^{\frac{1}{p-2}}$ converge when $p \to 2$.

u_p stays away from 0

Lemma

The family $(u_p)_{p>2}$ *stays away from* 0.

- $u_p \in \mathcal{M}_p \to \exists v_p \in \mathcal{N}_p$ such that $\int_\Omega v_p e_1 = 0$ and $\|v_p\| \leqslant \|u_p\|$
- By using the Hölder inequality, $\lambda_2 \int_\Omega v_p^2 \leqslant \int_\Omega |\nabla v_p|^2$ and Sobolev's inequalities, $(v_p)_{p>2}$ stays away from 0 ($\|v_p\|_p \geq (S\lambda_2^{-1})^{\frac{2^*}{2^*-2}\frac{1}{p}}$).



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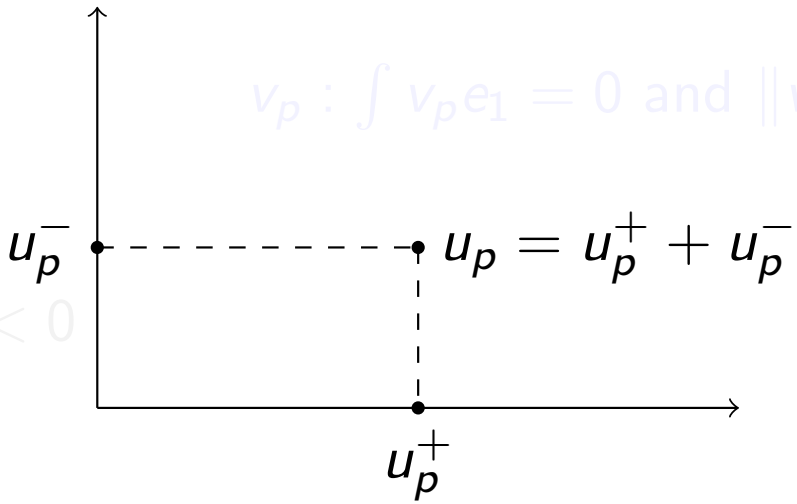

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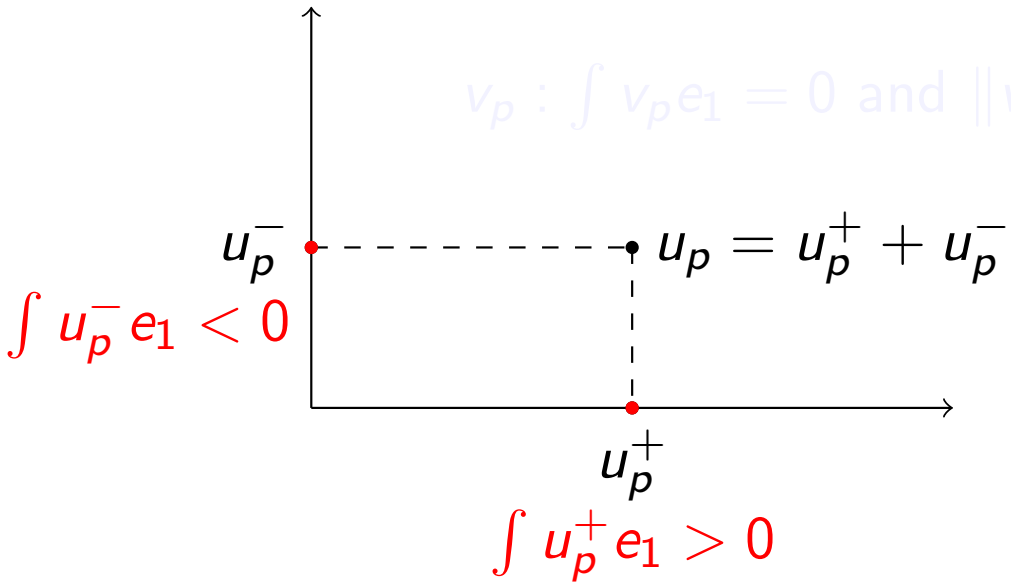

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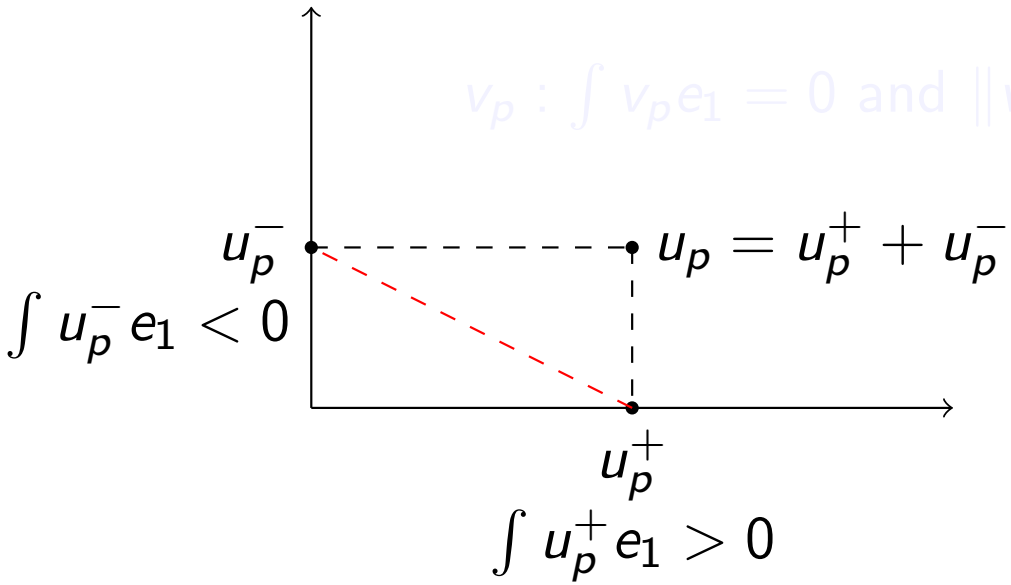

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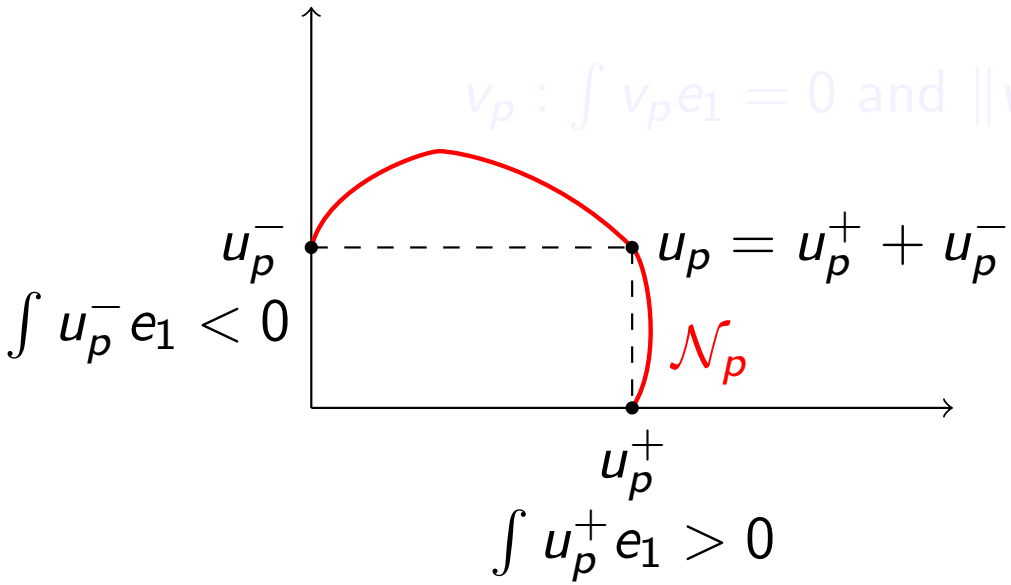

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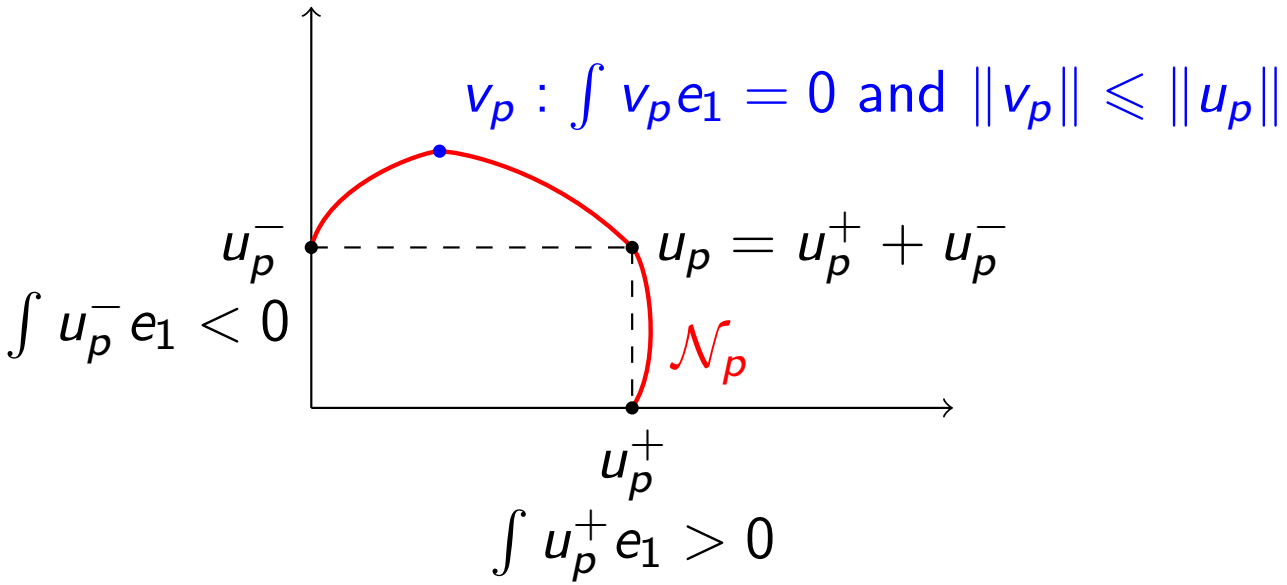

Using IFT

Let the problem (P_λ) $\begin{cases} \Delta u + u + \lambda |u|^{p-2} u = 0, & \text{in } \Omega \\ \mathrm{d}u = 0, & \text{in } \partial\Omega \\ \|u\| = 1 \end{cases}$

$$\varphi : [2, 2^*[\times H^1 \times \mathbb{R} \to H^1 \times \mathbb{R}$$
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Roots of φ are the solutions of (P_λ).

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General domains: sketch of the proof

The IFT approach does not work. This new approach is a **general approch** (but we lose the unicity!!!). Let $M > 0$, $\exists \varepsilon, \bar{p}_M > 0$,

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Sketch: Let work by contradiction, fix $w = P_{E_1} u$ and $z = P_{(E_1+E_2)^\perp} u$.

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Unicity at projection fixed in E_2

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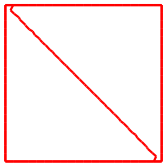
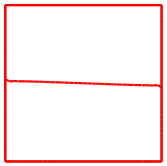
Idea: equation verified by $u_p - v_p$ (Δ linear) + Lebesgue
Corollary: If (u_p) is a family of bounded solutions staying away from 0, for p close to 2, u_p respect symmetries of its projection on E_2.

Square

- second eigenfunctions are odd with respect to the center $\rightarrow$ lens odd.





Accumulation points u_* verify
$J_*(u_*) = \inf\{J_*(u) : u \in E_2 \setminus \{0\}, \partial J_*(u)u = 0\}$, where
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Idea: $0 = \lim_{p\rightarrow 2}\int_\Omega \frac{(|u_p|^{p-2}u_p - u_p)v}{p-2} = \int_\Omega u_* log|u_*|v$.

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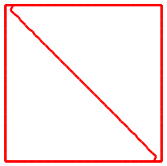

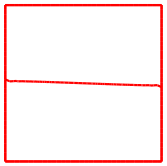

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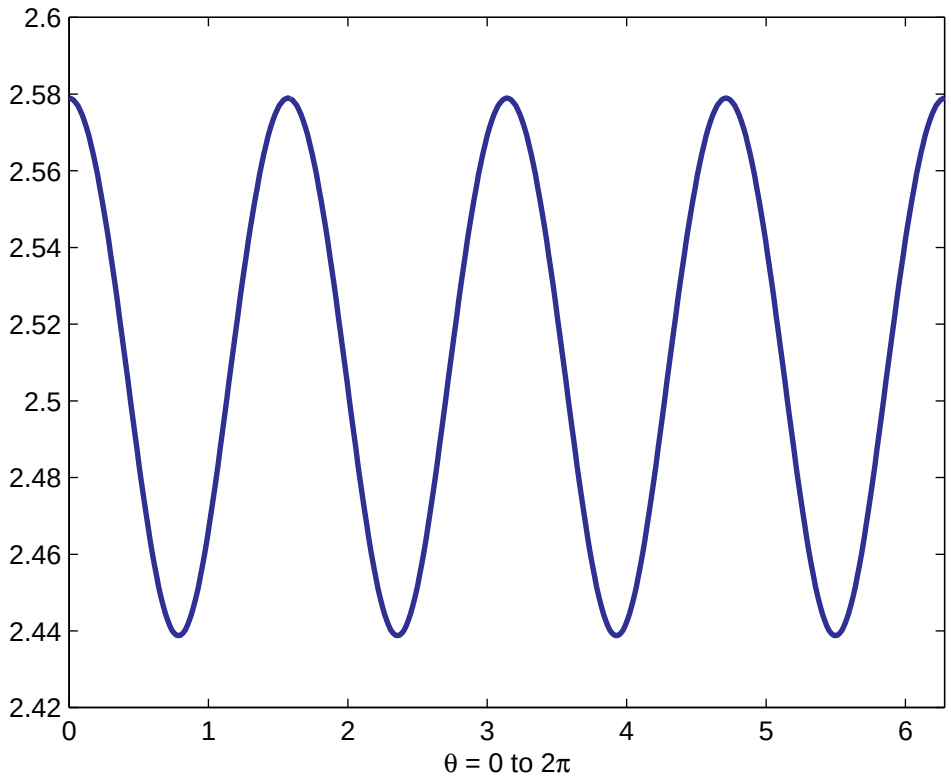
Limit functional on the limit Nehari manifold

J_* has Mountain-Pass structure $\rightarrow$ projection on the limit Nehari manifold.



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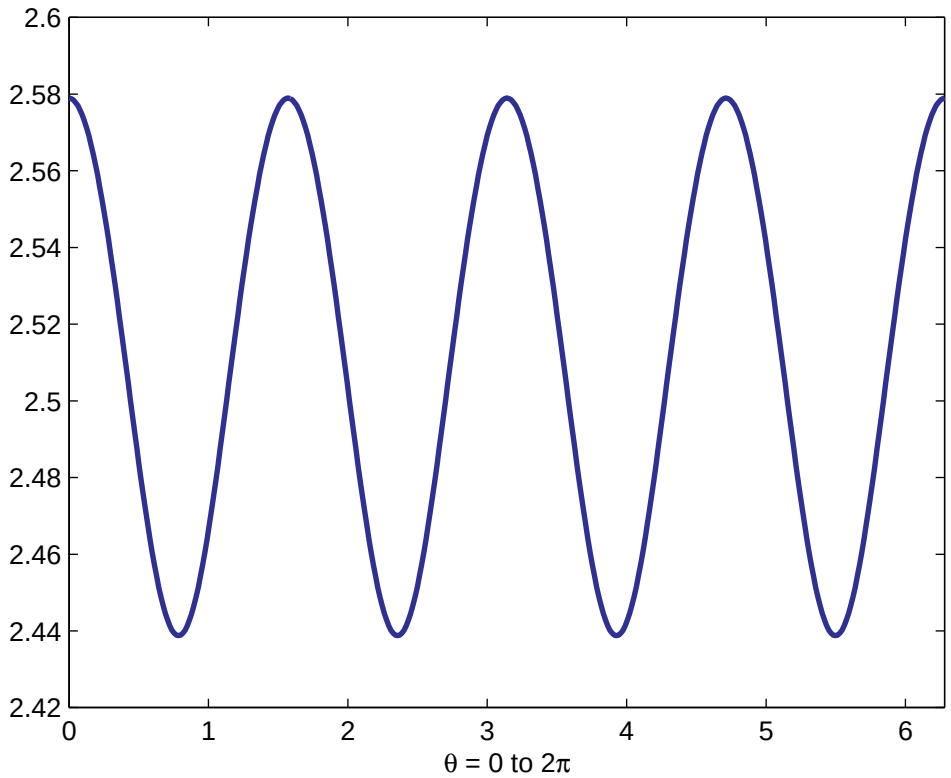

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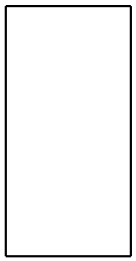
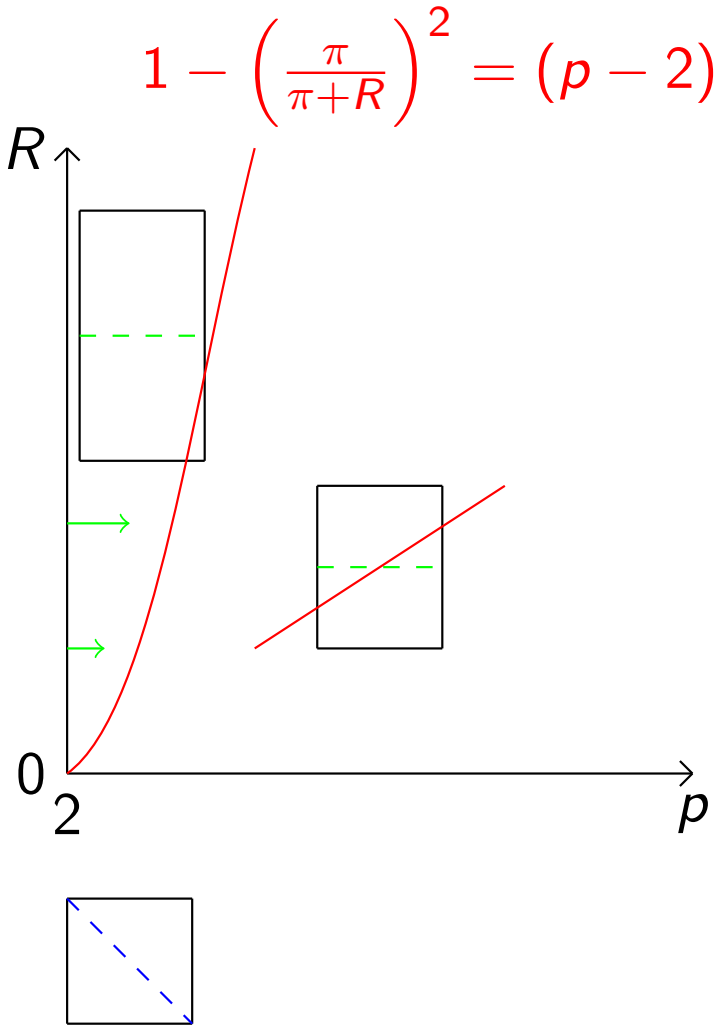
Symmetry breaking

wide $= \pi$



length $= \pi + R$



Idea: Results work for $-div(A_p \nabla u) + u = \lambda_2 |u|^{p-2} u$

Come back to ground state: summary

$$\begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } \Omega, \\ \mathrm{d}u = 0, & \text{on } \partial\Omega, \end{cases}$$

- For p close to 2, ground state solutions respect symmetries of its projection in E_1.
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As $1 < \lambda_2$, $w = 0$ and $v = 1$.

Improvement: control in L^1, L^∞, H^1, $W^{1,r}$-norm for any sequence of positive solution.

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Control for different norms

- By integration, $\int u_p = \int u_p^{p-1}$
- By interpolation, $\|u_p\|_{L^1} = \int u_p \leq |\Omega|$
- By Ni and Takagi, $\|u_p\|_H$ can be continuously controled by $\|u_p^{p-1}\|_{L^1}$ on each compact set of $]2, 2^*[$.
- By bootstrap, $\|u_p\|_H$ does not blow up when $p \to 2$
- By similar arguments, we obtain the control in $W^{1,r}$

So, 1 is the unique non-zero positive solution.

Schedule

1. p small?
2. p large?
   - Ground state are not symmetric,
   - bifurcation results,
   - multiplicity results.

1 not gs for p large and bifurcations: Ideas

1. $ind_M(1, \mathcal{E}_p)$ is the sum of dimensions of eigenspaces with negative eigenvalues λ of problem
$$\begin{cases} -\Delta h + h = (p-1)h + \lambda h, & \text{in}\Omega, \\ \mathrm{d}h = 0, & \text{on } \partial\Omega. \end{cases}$$
2. It corresponds to eigenvalues of $-\Delta + \mathrm{id}$ less than $p-1$.
3. So, for $\lambda_i < p-1 < \lambda_{i+1}$, $ind_M(1, \mathcal{E}_p) = \sum_{k \le i} \dim E_k$.
4. 1 is not ground state for $p > 1 + \lambda_2$.

Ground state is not radial on **radial domains**

For $p > 1 + \lambda_2$, ground state is **not** radial.

Sketch: By adapting results of A. Aftalion and F. Pacella,

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CRAS, 339(5), '04

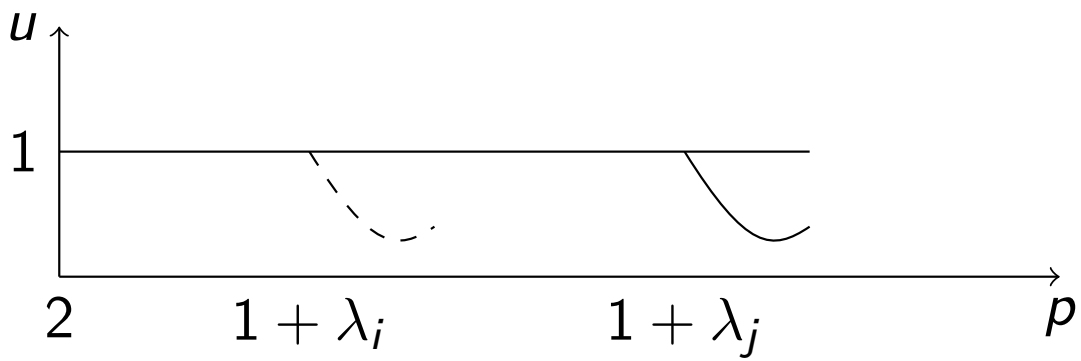
Morse index of not constant one-signed radial solution equals **at least** $N + 1$.
On general domains, a non-constant positive solution even w.r.t. n orthogonal hyperplanes and **constant** on the boundary cannot be a ground state

Other positive solutions ? Bifurcation results

Lemma

When $1 < p$ is increasing,

1. *Bifurcation* **sequence** *starting from* 1 **iff** p *crosses* $1 + \lambda_i$.
2. *Bifurcation branches if* λ_i *has* **odd** *multiplicity.*



Sketch of the proof

1. Problem **(P)** equivalent at $L(q,v) - N(q,v) = 0$, where
$$L : (1, 2^* - 1) \times H^1 \to H^1 : (q,v) \mapsto -q^{-1}v + (-\Delta + \mathrm{id})^{-1}v,$$
$$N : (1, 2^* - 1) \times H^1 \to H^1 : (q,v) \mapsto$$
$$(-\Delta + \mathrm{id})^{-1}(-v - q^{-1} + q^{-1}|1+v|^{q-1}(1+v)),$$
with $u = v + 1$ and $p = q + 1$.
2. Krasnoselskii-Bhoeme-Marino bifurcation result $\to$ bifurcations when $q = \lambda_i$
3. $\partial_v(L - N)(q, 0)$ invertible if $q \neq \lambda_i \to$ (by **IFT**) **no** bifurcation.

Remark: Branches also exist in $W^{1,r}$

KBM theorem: If $L = \lambda id - T$ with T linear, T and N completely continuous and $N = o(u)$ uniformly on each compact set of λ, $(\lambda, 0)$ is a global bifurcation point for λ an eigenvalue of T with odd multiplicity.

Properties of solutions along bifurcations (radial domains)

- radiality
- positivity
- unbounded

Radial bifurcations?

Idea: Do the same in H_{rad}.

- $E_i \cap H_{\text{rad}} = \{0\} \rightarrow \partial_v (L - N)_{H_{\text{rad}}}(\lambda_i, 0)$ invertible. So, **no radial** bifurcation.
- $E_i \cap H_{\text{rad}} \neq \{0\} \rightarrow$ at least one radial bifurcation.

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Positive bifurcations

Radial bifurcation ($X = W^{1,r}$ for r large)

- the bifurcation stays positive for a while.
- If **not** all the time, at the limit, the radial part of the solution possesses a non-simple zero (impossible by Höpf's lemma).

Other possibility:

$$L : (1, 2^* - 1) \times X \to X : (q, v) \mapsto -q^{-1}v + (-\Delta + \mathrm{id})^{-1}v,$$
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Non-crossing and unbounded

Let N_i be the number of zeros of radial part of functions in E_i.
Idea: Radial bifurcations starting from $1 + \lambda_i$ are included to
$S_i = \{$ functions with radial part possessing N_i intersections with $1\}$.

- It's working close to $(1, 1 + \lambda_i)$:
  - If not, there exists $(v_n, p_n) \to (1, 1 + \lambda_i)$ with $1 \neq v_n \notin S_i$.
  - $(v_n - 1, p_n - 1)$ solves $L - N = 0$.
  - For r large, $\frac{v_n - 1}{\|v_n - 1\|_{W^{1,r}}} \to h \neq 0$ up to a subseq ($W^{1,r} \subseteq C^{0,\alpha}$ and $C^{2,\alpha} \subseteq W^{1,r}$ (compact))
  - Moreover, $h \in E_i$ which is impossible as $v_n - 1$ just possesses simple zeros.
- It's working all the time: otherwise, we construct a solution with non-zero simple zeros.
- Unbounded: By Rabinowitz, bifurcation branches of $L - N = 0$ are linked by pair or unbounded

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Multiplicity

Theorem (D. Bonheure, G., preprint)

- *In dimension 2, for any $n \in \mathbb{N}_0$, there exists $\bar{p} > 2$ such that, for any $p > \bar{p}$, at least n positive solutions exist*
- *In large dimension, for any $2 < p < 2^*$ and $n \in \mathbb{N}_0$, at least n different positive solution exist if the measure of Ω is large enough.*

Idea: The blow-up given in the previous slide happens for $p > 2^*$.

Open questions

Questions?

- What about $p > 2^*$?
- General domains? (use of critical groups?)
- Is bifurcation given for $p = 1 + \lambda_2$ a ground state solutions branch?
- Is the first symmetry breaking occur for $\bar{p} = 1 + \lambda_2$?

Numerical experiments

By using Mountain-Pass algorithm:

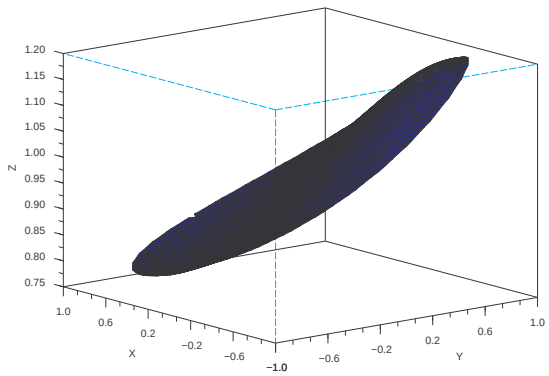

Figure: $-\Delta u + u = |u|^{\lambda_2 - 1 + 0.1} u$, with $\lambda_2 = 2 + \frac{\pi^2}{4}$

$\min u$	$\max u$	$\mathcal{E}_p(u)$	$\mathcal{E}_p(1)$
0.8	1.2	0.9	1.0

Table: Characteristics

Thank you for your attention