

ASYMPTOTICS AND SYMMETRIES OF LEAST ENERGY NODAL SOLUTIONS OF LANE–EMDEN PROBLEMS WITH SLOW GROWTH

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ABSTRACT. In this paper, we consider the Lane–Emden problem

$$\begin{cases} -\Delta u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^N and $p > 2$. First, we prove that, for p close to 2, the solution is unique once we fix the projection on the second eigenspace. From this uniqueness property, we deduce partial symmetries of least energy nodal solutions. We also analyze the asymptotic behaviour of least energy nodal solutions as p goes to 2. Namely, any accumulation point of sequences of (renormalized) least energy nodal solutions is a second eigenfunction that minimizes a reduced functional on a reduced Nehari manifold. From this asymptotic behaviour, we also deduce an example of symmetry breaking. We use numerics to illustrate our results.

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1. INTRODUCTION

We consider the super-linear elliptic boundary value problem

$$\begin{cases} -\Delta u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (\mathcal{P}_p)$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $N \geq 2$ and $p > 2$. It is well-known that if p is subcritical, i.e. $\frac{1}{p} > \frac{1}{2} - \frac{1}{N}$, Problem (\mathcal{P}_p) has a positive ground state solution [2]. B. Gidas, W. N. Ni and L. Nirenberg [8] showed, using the elegant and now celebrated moving planes technique, that, on a convex domain, the ground state inherits all the symmetries of the domain.

A. Castro, J. Cossio and J. M. Neuberger [4] proved the existence of a nodal solution with least energy among nodal solutions, which is therefore referred to as the *least energy nodal solution* of Problem (\mathcal{P}_p) . Since ground state solutions have the symmetries of the domain, a natural question is whether least energy nodal solutions inherit the symmetries of the domain Ω . In 2004, A. Aftalion and F. Pacella [1] proved that, on a ball, a least energy nodal solution cannot be radial. On the other hand, in 2005, T. Bartsch, T. Weth and M. Willem [3] obtained partial symmetry results: they showed that on a radial domain, a least energy nodal solution u has the so-called Schwarz foliated symmetry, i.e. u can be written as $u(x) = \tilde{u}(|x|, \xi \cdot x)$, where $\xi \in \mathbb{R}^N$ and $\tilde{u}(r, \cdot)$ is nondecreasing for every $r > 0$.

Motivated by [3], we study in this paper the question of the symmetry of least energy nodal solutions of Problem (\mathcal{P}_p) on more general domains. Indeed, the method of [3] fails when the group of the symmetries of the domain is discrete. As a first step, we study the problem when p is close to 2, taking a special care for specific domains.

In $H_0^1(\Omega)$, we work with the scalar product $(u, v) = \int_{\Omega} \nabla u \cdot \nabla v$. We denote by λ_i the i^{th} distinct eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, by E_i the i^{th} eigenspace, and, when λ_i is simple, by e_i an i^{th} eigenfunction such that $\|\nabla e_i\|_{L^2} = 1$. We denote by E_i^{\perp} the orthogonal space to E_i in $H_0^1(\Omega)$ and by P_{E_i} the orthogonal projection on E_i . We use the notation $\langle \cdot, \cdot \rangle$ for the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, $\|\cdot\|_{L^p}$ for the usual norm in $L^p(\Omega)$ and B_r to denote the closed ball $\{u \in H_0^1(\Omega) \mid \|u\| \leq r\}$.

Our first result basically shows that, for p close to 2, a priori bounded solutions of (\mathcal{P}_p) can be distinguished by their projections on the second eigenspace.

Proposition 1. *For every $M > 0$, there exists $\bar{p} > 2$ such that, for every $\alpha \in E_2$, for every $p \in]2, \bar{p}[$, Problem (\mathcal{P}_p) has at most one solution in the set $\{u \in B_M \mid P_{E_2}u = \alpha\}$.*

As readily expected, this uniqueness property immediately implies partial symmetries of least energy nodal solutions when the second eigenvalue λ_2 is simple, as for example on a rectangle, and p is close enough to 2.

Theorem 2. *Assume that λ_2 is simple. Then, for p close to 2 and any reflection R such that $R(\Omega) = \Omega$, least energy nodal solutions of Problem (\mathcal{P}_p) respect the symmetry or antisymmetry of e_2 with respect to R .*

In the particular case of a rectangle, we infer that the nodal line is the small median.

When λ_2 is not simple, the situation is more delicate. Indeed, one can already figure out the difficulties on a square as the second eigenfunctions do not necessarily have an axis of symmetry whereas one would expect so for a least energy nodal solution. When Ω is a ball, despite the degeneracy of the second eigenspace, we are still able to deduce a satisfactory statement.

Theorem 3. *Assume that Ω is a ball. Then, for p close to 2, least energy nodal solutions of Problem (\mathcal{P}_p) are radially symmetric with respect to $N - 1$ independent directions and antisymmetric with respect to the orthogonal one.*

The first part of this result holds in fact for any subcritical $p > 2$ as proved in [3]. To our knowledge, the antisymmetry property of the least energy nodal solution is not known in general. In particular, we obtain that the nodal line is a diameter. It should be pointed out that A. Aftalion and F. Pacella in [1] proved that, for a more general non-linearity, on a ball or an annulus in dimension two, the nodal line of a least energy nodal solution intersects the boundary of Ω .

The uniqueness property provided by Proposition 1 still allows to deduce a partial statement if all the second eigenfunctions enjoy some common symmetry. For instance, on a square, we deduce that least energy nodal solutions are antisymmetric with respect to the barycenter. In order to get further insight of symmetry properties of least energy nodal solutions, we study their asymptotic behaviour as $p \rightarrow 2$.

Theorem 4. *If $(u_p)_{p>2}$ are least energy nodal solutions of Problem (\mathcal{P}_p) , then*

$$\|\nabla u_p\|_{L^2} \leq C \lambda_2^{\frac{1}{p-2}}.$$

If $p_n \rightarrow 2$ and $\lambda_2^{\frac{1}{2-p_n}} u_{p_n} \rightarrow u_$ in $H_0^1(\Omega)$, then $\lambda_2^{\frac{1}{2-p_n}} u_{p_n} \rightarrow u_*$ in $H_0^1(\Omega)$, u_* satisfies*

$$\begin{cases} -\Delta u_* = \lambda_2 u_*, & \text{in } \Omega, \\ u_* = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$J_*(u_*) = \inf\{J_*(u) : u \in E_2 \setminus \{0\}, \langle dJ_*(u), u \rangle = 0\},$$

where

$$J_* : E_2 \rightarrow \mathbb{R} : u \mapsto \frac{\lambda_2}{2} \int_{\Omega} u^2 - u^2 \log u^2.$$

Beyond its own interest, Theorem 4 leads to the following conjecture.

Conjecture 5. *If Ω is a square and p is close to 2, least energy nodal solutions are symmetric with respect to a diagonal and antisymmetric in the orthogonal direction.*

Theorem 4 also highlights an example of symmetry breaking by carefully exploiting the degeneracy of the square and playing with the eccentricity of the rectangle. Namely, we prove the following result.

Theorem 6. *For every $p > 2$ sufficiently close to 2, there exists a rectangle Ω such that any least energy nodal solutions of Problem (\mathcal{P}_p) is neither symmetric nor antisymmetric with respect to the medians of Ω .*

The paper is organized as follows. We start with some preliminaries in Section 2. Proposition 1 is proved in Section 3 where we also work out an abstract symmetry result. We then proceed to the study of the asymptotics in Section 4 where Theorem 4 is proved. Section 5 deals with specific domains. It contains in particular the proofs of Theorem 2, Theorem 3 and numerical evidence of Conjecture 5. The proof of Theorem 6 relies on the study of a perturbation of Problem (\mathcal{P}_p) :

$$\begin{cases} -\operatorname{div}(A_p \nabla u) = |u|^{p-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (\tilde{\mathcal{P}}_p)$$

where A_p is a perturbation of the identity. This class of problems is analyzed in Section 6 where the proof of Theorem 6 is completed.

Throughout the paper, we use the Modified Mountain-Pass algorithm (MMPA) [6] to numerically illustrate our results. We point out that we are not aware of a proof of the convergence of this algorithm. Details on the computations are given in the appendix.

We close this introduction by pointing out that we could have worked with more general problems than Problem (\mathcal{P}_p) which plays the role of a paradigm of a superlinear problem

with slow growth. In particular, our arguments do not depend on the homogeneity of the nonlinear term and our approach can be extended to any family of superlinearities g_μ controlled by pure powers and such that $g_\mu(u) \rightarrow u$ as $\mu \rightarrow 0$ in a reasonable way.

2. PRELIMINARIES

By scaling, and to avoid further renormalizations, Problem (\mathcal{P}_p) can be rewritten as

$$\begin{cases} -\Delta u = \lambda_2 |u|^{p-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (\mathcal{P}_p)$$

2.1. Variational setting. Solutions of Problem (\mathcal{P}_p) are critical points of the energy functional J_p defined on $H_0^1(\Omega)$ by

$$J_p(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{\lambda_2}{p} \int_{\Omega} |u|^p.$$

The Nehari manifold \mathcal{N}_p and the nodal Nehari set \mathcal{M}_p are defined by

$$\mathcal{N}_p := \{u \in H_0^1(\Omega) \setminus \{0\} : \langle dJ_p(u), u \rangle = 0\}, \quad \mathcal{M}_p := \{u \in H_0^1(\Omega) : u^\pm \in \mathcal{N}_p\},$$

where $u^+(x) := \max(0, u(x))$ and $u^-(x) := \min(0, u(x))$. If $u \in H_0^1(\Omega)$, $u^+ \neq 0$ and $u^- \neq 0$, then $u \in \mathcal{M}_p$ if and only if

$$\int_{\Omega} |\nabla u^+|^2 = \lambda_2 \int_{\Omega} |u^+|^p \text{ and } \int_{\Omega} |\nabla u^-|^2 = \lambda_2 \int_{\Omega} |u^-|^p. \quad (1)$$

The interest of \mathcal{M}_p comes from the fact that it contains all the sign-changing critical points of J_p . If u minimizes J_p on \mathcal{M}_p then u is a nodal solution of (\mathcal{P}_p) usually referred to as the *least energy nodal solution*.

Theorem 2.1 (A. Castro, J. Cossio, J. M. Neuberger [4]). *There exists a least energy nodal solution of problem (\mathcal{P}_p) which has exactly two nodal domains.*

Let us notice that, for $u \in H_0^1(\Omega) \setminus \{0\}$, there exists a unique scalar $t_u > 0$ such that $t_u u \in \mathcal{N}_p$ and moreover the function $t_u u$ maximizes the energy functional in the direction of u .

2.2. Reduced problem. As mentioned in the introduction, see Theorem 4, we will consider the *reduced functional*

$$J_* : E_2 \rightarrow \mathbb{R} : u \mapsto \frac{\lambda_2}{2} \int_{\Omega} u^2 - u^2 \log u^2$$

(where $t^2 \log t^2$ is extended continuously by 0 at $t = 0$). The critical points of J_* are the functions u_* such that

$$\forall v \in E_2 : \int_{\Omega} v u_* \log u_*^2 = 0. \quad (2)$$

Any non-trivial critical points again belong to the *reduced Nehari manifold*

$$\mathcal{N}_* = \{u \in E_2 \setminus \{0\} : \langle dJ_*(u), u \rangle = 0\}.$$

This manifold is compact and such that $u \in \mathcal{N}_*$ if and only if

$$\int_{\Omega} u^2 \log u^2 = 0,$$

or equivalently if and only if

$$J_*(u) = \frac{\lambda_2}{2} \|u\|_{L^2}^2.$$

Observe that, for any $u \in E_2 \setminus \{0\}$, there exists again a unique constant $t_u^* > 0$ such that $t_u^* u \in \mathcal{N}_*$.

3. ABSTRACT SYMMETRY RESULT

In this section, we present an abstract symmetry result which is the basis to prove Theorem 2 and Theorem 3.

3.1. Uniqueness result. As discussed in the Introduction, for p close to 2, a priori bounded solutions of (\mathcal{P}_p) can be distinguished by their projections on the second eigenspace. This will follow from Proposition 3.2 below. We begin with a standard preliminary lemma. We provide a proof for completeness.

Lemma 3.1. *Let $N \geq 3$. There exists $\varepsilon > 0$ such that if $\|a(x) - \lambda_2\|_{L^{N/2}} < \varepsilon$ and u solves the boundary value problem*

$$\begin{cases} -\Delta u = a(x)u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

then either $u = 0$ or $P_{E_2}u \neq 0$.

Proof. Assume by contradiction that there exists a nontrivial solution u such that $P_{E_2}u = 0$. Let $w = P_{E_1}u$ and $z = P_{(E_1+E_2)^\perp}u$. Taking successively w and z as test functions and using Poincaré and Sobolev's inequalities, we infer that

$$\begin{aligned} \|\nabla w\|_{L^2}^2 &= \lambda_2 \int_{\Omega} w^2 + \int_{\Omega} (a(x) - \lambda_2)uw \\ &\geq \frac{\lambda_2}{\lambda_1} \|\nabla w\|_{L^2}^2 - C \|a(x) - \lambda_2\|_{L^{\frac{N}{2}}} \|\nabla w\|_{L^2} \|\nabla u\|_{L^2}, \\ \|\nabla z\|_{L^2}^2 &= \lambda_2 \int_{\Omega} z^2 + \int_{\Omega} (a(x) - \lambda_2)uz \\ &\leq \frac{\lambda_2}{\lambda_3} \|\nabla z\|_{L^2}^2 + C \|a(x) - \lambda_2\|_{L^{\frac{N}{2}}} \|\nabla z\|_{L^2} \|\nabla u\|_{L^2}. \end{aligned}$$

We deduce that

$$\begin{aligned} \|\nabla w\|_{L^2} &\leq \frac{\lambda_1 C}{\lambda_2 - \lambda_1} \|a(x) - \lambda_2\|_{L^{\frac{N}{2}}} \|\nabla u\|_{L^2}, \\ \|\nabla z\|_{L^2} &\leq \frac{\lambda_3 C}{\lambda_3 - \lambda_2} \|a(x) - \lambda_2\|_{L^{\frac{N}{2}}} \|\nabla u\|_{L^2}. \end{aligned}$$

Since $P_{E_2}u = 0$, we now conclude that

$$\|\nabla u\|_{L^2}^2 = \|\nabla w\|_{L^2}^2 + \|\nabla z\|_{L^2}^2 \leq C \|a(x) - \lambda_2\|_{L^{\frac{N}{2}}}^2 \|\nabla u\|_{L^2}^2.$$

When $\|a(x) - \lambda_2\|_{L^{\frac{N}{2}}}$ is small enough, this leads to a contradiction, so that the conclusion follows. \square

Observe that, if $N = 2$, the same statement can be formulated with the $L^{\frac{N}{2}}$ -norm replaced by any L^q -norm with $1 < q < \infty$. We only need to replace in the proof the use of the Sobolev inequality by the imbedding in any L^p with $1 < p < \infty$.

As a byproduct of the previous lemma, we deduce the following proposition.

Proposition 3.2. *For every $M > 0$, there exists $\bar{p} > 2$ such that, for every $p \in]2, \bar{p}[$, if $u_p \in B_M$ and $v_p \in B_M$ solve the boundary value problem*

$$\begin{cases} -\Delta u = \lambda_2 |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

then either $u_p = v_p$ or $P_{E_2}u_p \neq P_{E_2}v_p$.

Proof. Let $p_n \rightarrow 2$, $(u_n)_n \subseteq B_M$ and $(v_n)_n \subseteq B_M$ be two sequences such that u_n and v_n solve Problem (\mathcal{P}_p) with $p = p_n$ and $P_{E_2}u_n = P_{E_2}v_n$.

Since u_n and v_n are bounded in H_0^1 , up to subsequences, there exist $\alpha, \beta \in H_0^1(\Omega)$ such that $u_n \rightharpoonup \alpha$ and $v_n \rightharpoonup \beta$ in $H_0^1(\Omega)$. By assumption $P_{E_2}\alpha = P_{E_2}\beta$. By Rellich's Theorem, $u_n \rightarrow \alpha$ and $v_n \rightarrow \beta$ in $L^q(\Omega)$ for every $q < 2^*$. Therefore, $\alpha, \beta \in E_2$, so that $\alpha = \beta$.

Observe that

$$\begin{cases} -\Delta(u_n - v_n) = a_n(x)(u_n - v_n), & \text{in } \Omega, \\ u_n - v_n = 0, & \text{on } \partial\Omega, \end{cases} \quad (3)$$

where

$$a_n(x) = \begin{cases} \lambda_2 \frac{|u_n(x)|^{p_n-2}u_n(x) - |v_n(x)|^{p_n-2}v_n(x)}{u_n(x) - v_n(x)}, & \text{if } u_n(x) \neq v_n(x), \\ \lambda_2(p_n - 1)|u_n(x)|^{p_n-2}, & \text{if } u_n(x) = v_n(x). \end{cases}$$

Noting that

$$\frac{|t|^{p-2}t - |s|^{p-2}s}{t - s} \leq (p-1)(|t|^{p-2} + |s|^{p-2}),$$

we can apply Lebesgue's dominated convergence theorem, implying that $a_n \rightarrow \lambda_2$ in $L^q(\Omega)$, for every $q < \infty$. In particular, for every $\varepsilon > 0$ and n large enough, we have

$$\|a_n(x) - \lambda_2\|_{L^{\frac{N}{2}}} < \varepsilon.$$

Since $P_{E_2}(u_n - v_n) = 0$, Lemma 3.1 implies $u_n = v_n$ for large n . This concludes the proof. \square

Remark 3.3. It could seem more natural to apply the Implicit Function Theorem to prove Proposition 3.2 as done for example in [11] where the authors deal with a super-linear term with slow growth close to the first eigenvalue. However, the application of the Implicit Function Theorem to our situation is more delicate than in [11] due to a lack of smoothness in p . Hence, we preferred a direct argument which leads to a weaker conclusion but is enough to our purpose.

3.2. Abstract symmetry. We now apply the previous uniqueness result to deduce partial symmetries of the least energy nodal solution of Problem (\mathcal{P}_p) when p is close to 2. The next Lemma is the key ingredient.

Lemma 3.4. *Let $\alpha \in E_2 \setminus \{0\}$, $M > 0$ and \bar{p} be given by Proposition 3.2. If u_p is a solution of Problem (\mathcal{P}_p) such that $P_{E_2}u_p = \alpha$ and $u_p \in B_M$, $p < \bar{p}$, and if $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is a continuous isomorphism satisfying the conditions*

- (i) $T(E_2) = E_2$,
- (ii) $T(E_2^\perp) = E_2^\perp$,
- (iii) $T\alpha = \alpha$,
- (iv) for every $u \in H_0^1(\Omega)$, $J_p(Tu) = J_p(u)$,

then $Tu_p = u_p$.

Proof. By Proposition 3.2, it is sufficient to prove that Tu_p is a solution of Problem (\mathcal{P}_p) and $P_{E_2}Tu_p = \alpha$. On the one hand, it follows from (iv) that, for all $v \in H_0^1(\Omega)$, $\langle dJ_p(Tu_p), v \rangle = \langle dJ_p(u_p), T^{-1}v \rangle$, so that we infer that Tu_p is a solution of (\mathcal{P}_p) . On the other hand, since

$$Tu_p = TP_{E_2}u_p + TP_{E_2^\perp}u_p = P_{E_2}Tu_p + P_{E_2^\perp}Tu_p,$$

and the conditions (i) and (ii) ensure that $TP_{E_2}u_p \in E_2$ and $TP_{E_2^\perp}u_p \in E_2^\perp$, we deduce that $P_{E_2}Tu_p = TP_{E_2}u_p$. We then conclude the proof by using the condition (iii). \square

Let G be a group with identity 1. Recall that a group action on $H_0^1(\Omega)$ is a continuous application $G \times H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ such that, for all $u \in H_0^1(\Omega)$ and for all $g, h \in G$,

- (i) $1u = u$,
- (ii) $(gh)u = g(hu)$,
- (iii) $u \mapsto gu$ is linear.

Example 3.5. To any group $G \subseteq O(N)$ such that $g(\Omega) = \Omega$ for every $g \in G$, one can associate the action

$$gu(x) := u(gx).$$

Another action is given by

$$gu(x) := (\det g)u(gx).$$

When $G = \{1, R\}$, where R is the reflection with respect to a hyperplane H , the fixed points of the action of G are, in the first case, the symmetric functions with respect to H , and, in the second case, the antisymmetric functions with respect to H .

The next theorem is a straightforward consequence of Lemma 3.4.

Theorem 3.6. *Let $(G_\alpha)_{\alpha \in E_2}$ be groups acting on $H_0^1(\Omega)$ in such a way that, for every $g \in G_\alpha$ and for every $u \in H_0^1(\Omega)$,*

- (i) $g(E_2) = E_2$,
- (ii) $g(E_2^\perp) = E_2^\perp$,
- (iii) $g\alpha = \alpha$,
- (iv) $J_p(gu) = J_p(u)$.

Then, for all $M > 0$, if p is close enough to 2, any least energy nodal solutions $u_p \in B_M$ of Problem (\mathcal{P}_p) belongs to the fixator of G_{α_p} where $\alpha_p := P_{E_2}u_p$.

It is worth pointing out that in particular, if G_α describes the symmetries (or antisymmetries) of α , we deduce that, for p close to 2, u_p respects the symmetries of its orthogonal projection α_p . Specific cases will be discussed in Section 5.

4. ASYMPTOTIC BEHAVIOR

We now turn to the proof of Theorem 4. We proceed in several steps.

4.1. Upper bound. By a suitable choice of test functions, we first obtain an upper bound of the energy of least energy nodal solutions of Problem (\mathcal{P}_p) .

Lemma 4.1. *Let $(u_p)_{p>2}$ be a family of least energy nodal solutions of (\mathcal{P}_p) . Then we have*

$$\limsup_{p \rightarrow 2} \|\nabla u_p\|_{L^2}^2 = \limsup_{p \rightarrow 2} \left(\frac{J_p(u_p)}{\frac{1}{2} - \frac{1}{p}} \right) \leq \|\nabla u_*\|_{L^2}^2,$$

where $u_* \in E_2$ minimizes the functional J_* on \mathcal{N}_* .

Proof. Let $w \in H_0^1(\Omega)$ solves the problem

$$\begin{cases} -\Delta w - \lambda_2 w = \lambda_2 u_* \log |u_*|, \\ P_{E_2} w = 0. \end{cases} \quad (4)$$

Since u_* verifies (2), by the Fredholm alternative, w is well-defined. Set

$$v_p := u_* + (p-2)w$$

and

$$\hat{v}_p := t_p^+ v_p^+ + t_p^- v_p^-, \quad \text{where } t_p^\pm = \left(\frac{\|\nabla v_p^\pm\|_{L^2}^2}{\lambda_2 \|v_p^\pm\|_{L^p}^p} \right)^{\frac{1}{p-2}},$$

so that $\hat{v}_p \in \mathcal{M}_p$. We claim that $t_p^\pm \rightarrow 1$. By definition of v_p , we have

$$\begin{aligned}
\int_{\Omega} |\nabla v_p^{\pm}|^2 &= \int_{\Omega} \nabla v_p \cdot \nabla v_p^{\pm} \\
&= \int_{\Omega} (\lambda_2 u_* + \lambda_2 (p-2)w + \lambda_2 (p-2)u_* \log |u_*|) v_p^{\pm} \\
&= \lambda_2 \int_{\Omega} (v_p + (p-2)u_* \log |u_*|) v_p^{\pm} \\
&= \lambda_2 \int_{\Omega} |v_p|^{p-2} v_p v_p^{\pm} \\
&\quad + \lambda_2 (p-2) \left(\int_{\Omega} \frac{v_p - |v_p|^{p-2} v_p}{p-2} v_p^{\pm} + \int_{\Omega} u_* \log |u_*| v_p^{\pm} \right).
\end{aligned}$$

Since we have

$$\begin{aligned}
\left| \frac{t - |t|^{p-2}t}{p-2} \right| &= \frac{1}{p-2} \int_2^p |t \log |t|| |t|^{q-2} dq \\
&\leq |\log |t|| (|t| + |t|^{p-1}) \\
&\leq \frac{1}{s} (|t|^{1-s} + |t|^{p-1+s}),
\end{aligned}$$

where $s > 0$ has been chosen small enough, applying Lebesgue's dominated convergence theorem, we infer that

$$\lim_{p \rightarrow 2} \int_{\Omega} \left(\frac{v_p - |v_p|^{p-2} v_p}{p-2} + u_* \log |u_*| \right) v_p^{\pm} = 0.$$

Therefore, we deduce that

$$\int_{\Omega} |\nabla v_p^{\pm}|^2 = \lambda_2 \int_{\Omega} |v_p^{\pm}|^p + o(p-2), \tag{5}$$

so that $\lim_{p \rightarrow 2} t_p^{\pm} = 1$. At last, since

$$J_p(u_p) \leq J_p(\hat{v}_p)$$

and

$$\left(\frac{1}{2} - \frac{1}{p} \right) J_p(\hat{v}_p) = \|\nabla \hat{v}_p\|_{L^2}^2 = \|\nabla u_*\|_{L^2}^2 + o(1),$$

the conclusion follows easily. \square

Remark 4.2. When working with ground states on the Nehari manifold, one obtains an optimal upper bound by evaluating the functional on eigenfunctions that are in the Nehari manifold. This is not anymore the case for least energy nodal solutions. Such a choice leads to a coarser estimate:

$$\|\nabla u_*^+\|_{L^2}^2 \exp\left(\frac{-\int_{\Omega} |u_*^+|^2 \log |u_*^+|^2}{\int_{\Omega} |u_*^+|^2}\right) + \|\nabla u_*^-\|_{L^2}^2 \exp\left(\frac{-\int_{\Omega} |u_*^-|^2 \log |u_*^-|^2}{\int_{\Omega} |u_*^-|^2}\right),$$

This is why we define $v_p = u_* + (p-2)w$. Geometrically, this can be pictured as follows: the natural projection of u_* on the nodal Nehari set is far from u_* , but u_* gets nearer to the Nehari manifold as $p \rightarrow 2$.

4.2. Limit equation. In this Section, we will consider the weak accumulation points u_* of a bounded family $(u_p)_{p>2}$ of solutions of Problem (\mathcal{P}_p) , and prove that those functions verify a limit equation.

Lemma 4.3. *Let $(u_p)_{p>2}$ be a bounded family of solutions of (\mathcal{P}_p) . If $p_n \rightarrow 2$ and $u_{p_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$, then u_* solves*

$$\begin{cases} -\Delta u_* = \lambda_2 u_*, & \text{in } \Omega, \\ u_* = 0, & \text{on } \partial\Omega, \\ \int_{\Omega} u_* \log |u_*| v = 0, & \forall v \in E_2. \end{cases}$$

Proof. Let $v \in H_0^1(\Omega)$. By Rellich's theorem and Lebesgue's dominated convergence theorem, we deduce that

$$|u_{p_n}|^{p_n-2} u_{p_n} \rightharpoonup u_* \text{ in } L^2(\Omega),$$

so that

$$\begin{aligned} \int_{\Omega} \nabla u_* \cdot \nabla v &= \lim_{n \rightarrow \infty} \int_{\Omega} \nabla u_{p_n} \cdot \nabla v \\ &= \lim_{n \rightarrow \infty} \lambda_2 \int_{\Omega} |u_{p_n}|^{p_n-2} u_{p_n} v \\ &= \lambda_2 \int_{\Omega} u_* v. \end{aligned}$$

Henceforth, $u_* \in E_2$. To prove the second statement, taking $v \in E_2$ and multiplying the equation in (\mathcal{P}_p) by v lead to

$$\int_{\Omega} (|u_{p_n}|^{p_n-2} u_{p_n} - u_{p_n}) v = 0. \quad (6)$$

Finally, arguing as in the previous theorem, we conclude using Lebesgue's dominated convergence theorem and (4.1) that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{(|u_{p_n}|^{p_n-2} u_{p_n} - u_{p_n}) v}{p_n - 2} = \int_{\Omega} u_* \log |u_*| v.$$

Taking (6) into account, this completes the proof. \square

4.3. Lower bound. We now prove that least energy nodal solutions of problem (\mathcal{P}_p) stay away from 0 when $p \rightarrow 2$.

Lemma 4.4. *Let $(u_p)_{p>2}$ be least energy nodal solutions of Problem (\mathcal{P}_p) . Then we have*

$$\liminf_{p \rightarrow 2} \|\nabla u_p\|_{L^2} > 0.$$

Proof. For each p , we define the function v_p as follows. Let $r = \frac{\int_{\Omega} u_p^+ e_1}{\int_{\Omega} |u_p| e_1}$ and take

$$\tilde{v}_p = (1-r)u_p^+ + r u_p^-.$$

Our choice of r implies that \tilde{v}_p is orthogonal to e_1 . Choose now $t \geq 0$ in such a way that $v_p := t \tilde{v}_p \in \mathcal{N}_p$. By construction, $v_p \in E_1^\perp \cap \mathcal{N}_p$.

First, since $u_p \in \mathcal{M}_p$, we have

$$J_p(v_p) \leq J_p(u_p),$$

so that

$$\|\nabla v_p\|_{L^2} \leq \|\nabla u_p\|_{L^2}.$$

It is now sufficient to show that v_p is bounded away from 0 in $H_0^1(\Omega)$ when $p \rightarrow 2$.

By interpolation of Hölder's inequality between 2 and $q < 2^*$ with $p < q$, we obtain

$$\|v_p\|_{L^p}^2 \leq \|v_p\|_{L^2}^{2-2s} \|v_p\|_{L^q}^{2s},$$

where $s := \frac{q}{2-q} \frac{2-p}{p}$. On the other hand, as v_p is orthogonal to e_1 , Poincaré's inequality ensures that

$$\lambda_2 \int_{\Omega} v_p^2 \leq \int_{\Omega} |\nabla v_p|^2.$$

Therefore, by Sobolev's embedding, we obtain

$$\|v_p\|_{L^p}^2 \leq (\lambda_2^{-1} \|\nabla v_p\|_{L^2}^2)^{1-s} (C \|\nabla v_p\|_{L^2}^2)^s.$$

As v_p belongs to the Nehari manifold \mathcal{N}_p , we have

$$\|\nabla v_p\|_{L^2}^2 = \lambda_2 \|v_p\|_{L^p}^p.$$

It then follows that

$$\|\nabla v_p\|_{L^2} \geq (C \lambda_2)^{\frac{sp}{2(2-p)}} \lambda_2^{\frac{1}{2}}$$

and from our choice of s , we conclude that

$$\|\nabla v_p\|_{L^2} \geq (C \lambda_2)^{\frac{q}{2(2-q)}} \lambda_2^{\frac{1}{2}}. \quad \square$$

4.4. Conclusion. Bringing together the previous lemmas, we now deduce

Theorem 4.5. *Let $(u_p)_{p>2}$ be least energy nodal solutions of Problem (\mathcal{P}_p) . If $p_n \rightarrow 2$ and $u_{p_n} \rightarrow u_*$ in $H_0^1(\Omega)$, then $u_{p_n} \rightarrow u_*$ in $H_0^1(\Omega)$, u_* is such that*

$$\begin{cases} -\Delta u_* = \lambda_2 u_*, & \text{in } \Omega, \\ u_* = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$J_*(u_*) = c := \inf\{J_*(u) : u \in E_2 \setminus \{0\}, \langle dJ_*(u), u \rangle = 0\}.$$

Proof. Observe first that

$$\inf\{\|\nabla u\|_{L^2}^2 : u \in E_2 \setminus \{0\}, \int_{\Omega} u^2 \log u^2 = 0\} = 2c$$

and by Lemma 4.3, $u_* \in E_2$. Applying successively Lemma 4.1, the weak lower semi-continuity of the norm and Lemma 4.3, we deduce that

$$2c \geq \limsup_{n \rightarrow \infty} \|\nabla u_{p_n}\|_{L^2}^2 \geq \liminf_{n \rightarrow \infty} \|\nabla u_{p_n}\|_{L^2}^2 \geq \|\nabla u_*\|_{L^2}^2 \geq 2c.$$

We hence conclude that $\lim_{n \rightarrow \infty} \|\nabla u_{p_n}\|_{L^2}^2 = \|\nabla u_*\|_{L^2}^2 = 2c$. This also implies immediately the strong convergence of the sequence $(u_{p_n})_n$. \square

5. SYMMETRIES ON SPECIFIC DOMAINS

The abstract symmetry result presented in Section 3 allows to prove that, if some symmetry is shared by all the second eigenfunctions, then it is inherited by the solutions u_p .

5.1. Symmetries in the nondegenerate case.

Corollary 5.1. *If λ_2 is simple, for any reflection R such that $R(\Omega) = \Omega$ and p close to 2, least energy nodal solutions of Problem (\mathcal{P}_p) respect the symmetry or antisymmetry of e_2 with respect to R .*

Proof. Consider the first group action of Example 3.5 if $e_2 \circ R = e_2$ or the second one if $e_2 \circ R = -e_2$, and conclude by Theorem 3.6. \square

Corollary 5.1 can be illustrated numerically. Consider a rectangle of sidelengths 1 and 2. Figure 1 shows a nodal solution u of the problem $-\Delta u = u^3$ with Dirichlet boundary conditions obtained by the MMPA. While it is not proved that this solution has least energy, all the other nodal solutions that we have found numerically have a larger energy. Moreover, the Mountain-Pass Algorithm (MPA) suggests that u^+ and u^- are ground states of Problem (\mathcal{P}_p) on the squares defined by the nodal regions. One sees in Figure 2 that

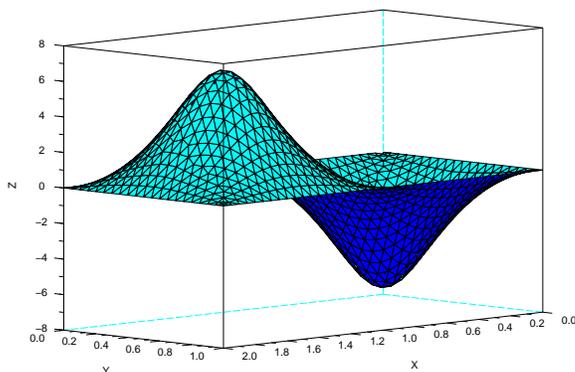
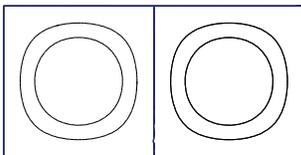


FIGURE 1. MPA solution of the Lane–Emden problem on a rectangle

FIGURE 2. Level lines at levels -1 , -0.5 , 0 , 0.5 and 1

u is antisymmetric with respect to the shortest median, which is thus also the nodal line. More examples can be found in [9].

5.2. Symmetries on the ball.

Corollary 5.2. *If Ω is a ball, for p close to 2, least energy nodal solutions of Problem (\mathcal{P}_p) are radially symmetric with respect to $N - 1$ independent directions and antisymmetric with respect to the orthogonal one.*

Proof. By classical separation of variables and properties of the zeroes of Bessel functions, any second eigenfunction can be written as $e(x) = R(|x|)(\xi, x)$, $\xi \in \mathbb{R}^N$ (see e.g. [10]). Therefore, up to rotation, it belongs to the fixator of the group $G = O(N - 1) \times \{1, -1\}$ acting as $gu(x', x_N) = ku(hx', kx_N)$, for $g := (h, k)$. Previous arguments allow to conclude. \square

Let us illustrate this on the unit ball in \mathbb{R}^2 . Figure 3 depicts a nodal solution u of the problem $-\Delta u = u^3$ with Dirichlet boundary conditions. As previously, comparison with the energy of other numerically computed nodal solutions of the Dirichlet problem and the application of the MPA to u^+ and u^- suggest that the algorithm indeed caught a least energy nodal solution. One should note that, as shown by A. Aftalion and F. Pacella, u is not radial, u is symmetric with respect to a direction as proved by T. Bartsch, T. Weth and M. Willem, see [3], and, as suggested by Corollary 5.2, antisymmetric in the orthogonal one. In particular, the nodal line is a diameter.

5.3. Symmetries on the square.

Corollary 5.3. *If Ω is a square, then, for p close to 2, least energy nodal solutions of Problem (\mathcal{P}_p) are odd with respect to the center of the square.*

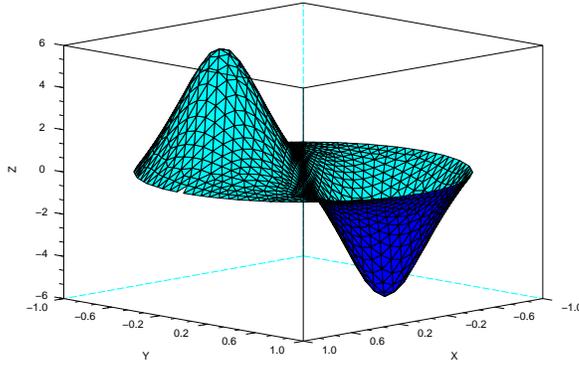
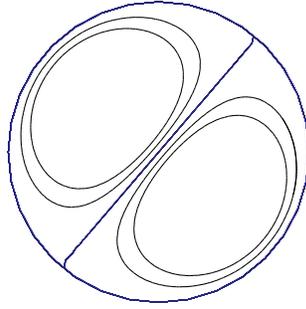


FIGURE 3. MPA solution of the Lane–Emden problem on a ball

FIGURE 4. Level lines at levels -1 , -0.5 , 0 , 0.5 and 1

Proof. Without loss of generality, we can work on square $[-1, 1]^2$. The functions

$$v_1(x, y) = \cos\left(\frac{\pi}{2}x\right) \sin(\pi y), \quad v_2(x, y) = \sin(\pi x) \cos\left(\frac{\pi}{2}y\right),$$

form an orthonormal basis of E_2 . One checks directly that all the second eigenfunctions of $-\Delta$ are odd functions. They all belong to the fixator of the group $G := \{1, -1\}$ acting in such a way that $(-1)u(x) = -u(-x)$. One concludes by Theorem 3.6. \square

A nodal solution u of the problem $-\Delta u = u^3$ with Dirichlet boundary conditions on the square $[-1, 1]^2$ is depicted in Figure 5. As explained in the previous cases, it is expected that u is a least energy nodal solution. Figure 6 suggests that u is odd.

Moreover, the nodal line of u seems to be a diagonal, u is antisymmetric with respect to that diagonal and symmetric with respect to the other diagonal. Further numerical computations confirm our guess. Indeed, for $\theta \in \mathbb{R}$, set

$$v_\theta = \cos \theta v_1 + \sin \theta v_2$$

and consider the function

$$E_*(\theta) = \sup_{t>0} J_*(tv_\theta).$$

Since $\int_\Omega |v_\theta|^2 = 1$, one easily computes

$$S(\theta) := \log \frac{2}{\lambda_2} E_*(\theta) = - \int_\Omega |v_\theta|^2 \log |v_\theta|^2.$$

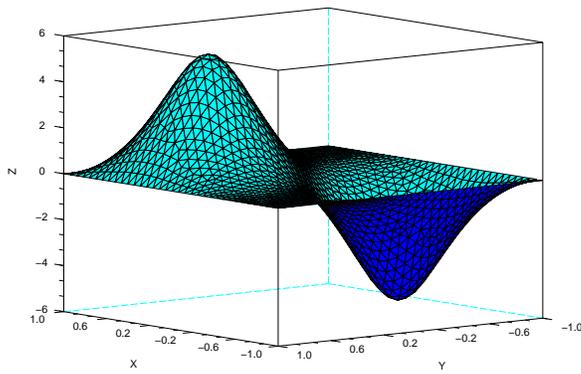


FIGURE 5. MPA solution of the Lane–Emden problem on a square

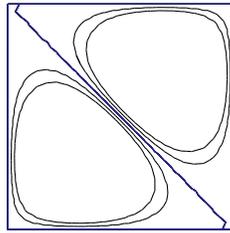


FIGURE 6. Level lines at levels $-1, -0.5, 0, 0.5$ and 1

This function can be thought as the entropy associated to the density $|v|^2$, and the accumulation points of the minimal energy nodal solutions are the eigenfunctions with minimal entropy. The numerical computation of S , see Figure 7, suggests that these eigenfunctions

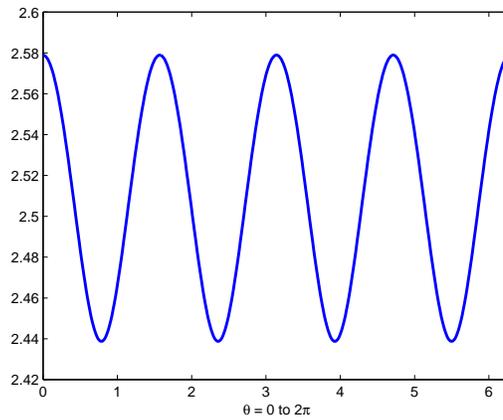


FIGURE 7. The function S

are the functions v_θ with $\theta = \frac{\pi}{4} + k\frac{\pi}{2}$. Up to rotations of $k\frac{\pi}{2}$, these can be written as

$$4 \cos \frac{\pi}{2}x \cos \frac{\pi}{2}y \cos \frac{\pi}{2}(x - y) \sin \frac{\pi}{2}(x + y).$$

These computations legitimate the following conjecture.

Conjecture 5.4. If Ω is a square, then, for p close to 2, any least energy nodal solution of Problem (\mathcal{P}_p) is symmetric with respect to one diagonal and antisymmetric with respect to the orthogonal one.

6. SYMMETRY BREAKING

In the previous section, the study of accumulation points of least energy nodal solutions $(u_p)_{p>2}$ allowed to exhibit the symmetries of least energy nodal solutions for p close to 2. This analysis can also be used to prove some symmetry breaking. We begin by studying small perturbations of the Laplacian.

6.1. Perturbation of the Laplacian operator. The previous results can be extended to the problem

$$\begin{cases} -\operatorname{div}(A_p \nabla u) = \lambda_2 |u|^{p-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (\tilde{\mathcal{P}}_p)$$

with $A_p \in \mathcal{C}(\Omega, S^{N \times N})$, where $S^{N \times N}$ is the set of symmetric $N \times N$ matrices, such that $A_2 = id$ and $p \mapsto A_p$ is uniformly differentiable at $p = 2$, i.e.

$$\left\| \frac{A_p - A_2 - A'_2(p-2)}{p-2} \right\|_{\infty} \rightarrow 0$$

as $p \rightarrow 2$.

We are interested in critical points of the functional

$$\tilde{J}_p : H_0^1(\Omega) \rightarrow \mathbb{R} : u \mapsto \frac{1}{2} \int_{\Omega} (A_p \nabla u) \cdot \nabla u - \frac{\lambda_2}{p} \int_{\Omega} |u|^p.$$

Theorem 6.1. Let $(u_p)_{p>2}$ be least energy nodal solutions of Problem $(\tilde{\mathcal{P}}_p)$. If $p_n \rightarrow 2$ and $u_{p_n} \rightharpoonup u_*$ in $H_0^1(\Omega)$, then $u_{p_n} \rightarrow u_*$ in $H_0^1(\Omega)$, u_* satisfies

$$\begin{cases} -\Delta u_* = \lambda_2 u_*, & \text{in } \Omega, \\ u_* = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$\tilde{J}_*(u_*) = \inf\{\tilde{J}_*(u) : u \in E_2 \setminus \{0\}, \langle dJ_*(u), u \rangle = 0\},$$

where

$$\tilde{J}_* : E_2 \rightarrow \mathbb{R} : u \mapsto \int_{\Omega} A'_2 \nabla u \cdot \nabla u + \frac{\lambda_2}{2} (u^2 - u^2 \log u^2).$$

Sketch of the proof. All the arguments developed in Section 4.1 hold for $(\tilde{\mathcal{P}}_p)$, up to some small differences. Indeed, the equations (4) and (5) become respectively

$$\begin{cases} -\Delta w - \lambda_2 w = \operatorname{div}(A'_2 \nabla u_*) + \lambda_2 u_* \log |u_*|, \\ P_{E_2} w = 0, \end{cases}$$

and

$$\begin{aligned} \int_{\Omega} |\nabla v_p^{\pm}|^2 &= \lambda_2 \int_{\Omega} (v_p + (p-2)u_* \log |u_*|) v_p^{\pm} - (p-2) \int_{\Omega} (A'_2 \nabla u_*) \cdot \nabla v_p^{\pm} \\ &= \lambda_2 \int_{\Omega} |v_p^{\pm}|^p - (p-2) \int_{\Omega} (A'_2 \nabla u_*) \cdot \nabla v_p^{\pm} + o(p-2). \end{aligned}$$

The computation of t_p^{\pm} gives here

$$(t_p^{\pm})^{p-2} = \frac{\|A_p \nabla v_p^{\pm}\|_{L^2}^2}{\lambda_2 \|v_p^{\pm}\|_{L^p}^p} = \frac{\|\nabla v_p^{\pm}\|_{L^2}^2 + (p-2) \int_{\Omega} \left(\frac{A_p - id}{p-2} \nabla v_p\right) \cdot \nabla v_p^{\pm}}{\lambda_2 \|v_p^{\pm}\|_{L^p}^p},$$

and the proof that $t_p^\pm \rightarrow 1$ follows from the fact that

$$\begin{aligned} & (p-2) \int_{\Omega} \left(\frac{A_p - id}{p-2} \nabla v_p \right) \cdot \nabla v_p^\pm - (p-2) \int_{\Omega} (A'_2 \nabla u_*) \cdot \nabla v_p^\pm \\ &= (p-2) \int_{\Omega} \left(\left(\frac{A_p - id}{p-2} - A'_2 \right) \nabla v_p \right) \cdot \nabla v_p^\pm + (p-2)^2 \int_{\Omega} (A'_2 \nabla w) \cdot \nabla v_p^\pm \\ &= o(p-2), \end{aligned}$$

giving then the same conclusion as for Problem (\mathcal{P}_p) .

In a similar fashion, all the results of Sections 4.2 and 4.3 extend to the present framework, so that we can conclude as in Section 4.4. \square

Going to abstract symmetry results, Proposition 3.2 about the uniqueness of the solution with a given projection on E_2 extends immediately to Problem $(\tilde{\mathcal{P}}_p)$.

Proposition 6.2. *For every $M > 0$, there exists $\bar{p} > 2$ such that, for every $p \in]2, \bar{p}[$, if $u_p \in B_M$ and $v_p \in B_M$ solve the boundary value problem*

$$\begin{cases} -\operatorname{div}(A_p \nabla u) = \lambda_2 |u|^{p-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

then either $u_p = v_p$ or $P_{E_2} u_p \neq P_{E_2} v_p$.

So, we obtain

Theorem 6.3. *Let $(G_\alpha)_{\alpha \in E_2}$ be groups acting on $H_0^1(\Omega)$ in such a way that, for every $g \in G_\alpha$ and for every $u \in H_0^1(\Omega)$,*

- (i) $g(E_2) = E_2$,
- (ii) $g(E_2^\perp) = E_2^\perp$,
- (iii) $g\alpha = \alpha$,
- (iv) $J_p(gu) = J_p(u)$.

Then, for p close to 2, any least energy nodal solutions u_p of Problem $(\tilde{\mathcal{P}}_p)$ belongs to the fixator of G_{α_p} where $\alpha_p := P_{E_2} u_p$.

6.2. Symmetry breaking on the rectangle. On a rectangle R_ν with sides of lengths 2 and $(1+\nu)2$, we consider the problem

$$\begin{cases} -\Delta u = \lambda_2 |u|^{p-2} u, & \text{in } R_\nu, \\ u = 0, & \text{on } \partial R_\nu. \end{cases} \quad (\mathcal{P}_\nu)$$

The change of variable $\tilde{u}(x, y) = u(x, (1+\nu)y)$, leads to the equivalent problem on the square $Q =]-1, 1[^2$

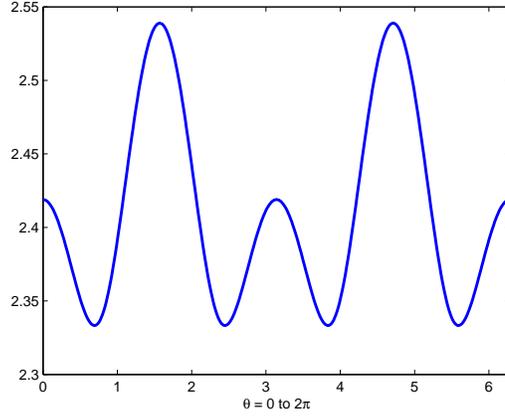
$$\begin{cases} -\partial_x^2 \tilde{u} - (1+\nu)^{-2} \partial_y^2 \tilde{u} = \lambda_2 |\tilde{u}|^{p-2} \tilde{u}, & \text{in } Q, \\ \tilde{u} = 0, & \text{on } \partial Q. \end{cases}$$

In both problems, $\lambda_2 = \lambda_2(Q)$. Observe also that u is a least energy nodal solution of (\mathcal{P}_ν) if and only if \tilde{u} is a least energy nodal solution on the square Q .

Theorem 6.4. *There exists $C > 0$ and \bar{p} such that if $2 < p < \bar{p}$ and $|\nu| \leq C(p-2)$, then every least energy nodal solution of problem (\mathcal{P}_ν) is neither symmetric nor antisymmetric with respect to one of the medians of R_ν .*

Proof. Assume by contradiction, that there exists $p_n \rightarrow 2$ and $\nu_n = o(p_n - 2)$ such that Problem (\mathcal{P}_{ν_n}) has a solution u_n that is symmetric or antisymmetric to one of the medians. Define \tilde{u}_n by the change of variables given above. The functions \tilde{u}_n are least energy nodal solutions of the problem $(\tilde{\mathcal{P}}_{p_n})$ with $\Omega = Q$ and

$$A_{p_n} = id - (1 - (1 + \nu_n)^{-2}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

FIGURE 8. $\gamma = 0.05$

The corresponding functional \tilde{J}_* reads as J_* for Problem (\mathcal{P}_p) . In particular, the accumulation points of the sequence are symmetric or antisymmetric with respect to one of the medians. On the other hand, numerical computations illustrated by Figure 7 shows that such eigenfunctions cannot be minimizers of \tilde{J}_* on $\tilde{\mathcal{N}}_*$ (one would indeed conjecture that they are maximizers on $\tilde{\mathcal{N}}_*$). \square

Now, consider the case where $\nu = \gamma(p - 2)$. Arguing as in section 5.3, we can compute

$$\tilde{E}(\theta) = \sup_{t > 0} \tilde{J}_*(t\theta)$$

and

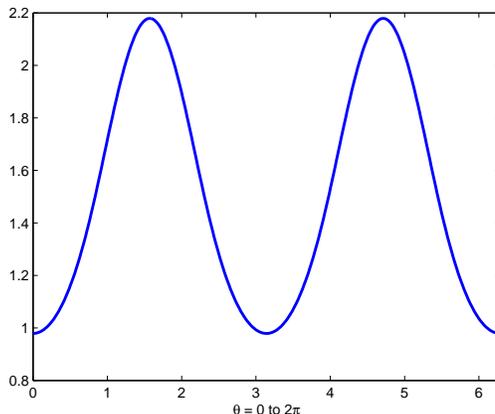
$$\begin{aligned} \tilde{S}_\gamma(\theta) &:= \log \frac{2}{\lambda_2} \tilde{E}(\theta) \\ &= S(\theta) - \frac{4\gamma}{\lambda_2} \int_Q (\cos \theta \partial_y v_1 + \sin \theta \partial_y v_2)^2 \\ &= S(\theta) - \frac{4\gamma}{5} (3 \cos^2 \theta + 1). \end{aligned}$$

The function S achieves a global minimum at $\frac{\pi}{4} + k\frac{\pi}{2}$, see Figure 7, while the function $\frac{-4\gamma}{5} (3 \cos^2 \theta + 1)$ achieves its global minimum at $k\pi$. Since these points are critical points of S , the global minimizers of $\tilde{S}_\gamma(\theta)$ are $k\pi$ when γ is large enough. The functions v_θ for these values of θ being symmetric with respect to the medians of Q , we naturally conjecture

Conjecture 6.5. There exists $c > 0$ and $\bar{p} > 2$, such that if $\nu > c(p - 2)$ and $2 < p < \bar{p}$, then every least energy nodal solutions to problem (\mathcal{P}_ν) is symmetric with respect to the longest median and antisymmetric with respect to the shortest one.

The shapes of the graphs of $\tilde{S}_{0.05}$ and $\tilde{S}_{0.5}$ (see Figure 8 and Figure 9) seem to indicate that the threshold value for γ corresponds to a degenerate minima. Hence, it is expected that the first γ for which $k\pi$ is a global minimum should verify $\tilde{S}_\gamma''(k\pi) = S''(k\pi) + \frac{24}{5}\gamma = 0$. A numerical approximation of the second derivative of S gives

$$\gamma = 0.2167.$$

FIGURE 9. $\gamma = 0.5$

APPENDIX A. NUMERICAL COMPUTATIONS

We have used the modified Mountain-Pass algorithm (MMPA) [6] to compute nodal solutions of problems with Mountain-Pass geometry. This algorithm is based on the Mountain-Pass algorithm (MPA) which computes one-signed solution of such problems and was introduced in 1993 by Y. S. Choi and P. J. McKenna [5]. In 2001, J. Zhou and Y. Li [12, 13] proved the convergence of a variant of the algorithm due to Y. S. Choi and P. J. McKenna.

The algorithm relies at each step on the finite element method (see e.g. [7]). The domain Ω is triangulated with a Delaunay condition in such a manner that the distance between two nodes on the boundary of Ω is 0.05. We use the Easymesh software to do it.

The program stops when the gradient of the energy functional at the approximations has a norm strictly inferior than 1.0×10^{-2} . We use the Java language to compute the MMPA and the Scilab software to graph numerical solutions. The starting functions are given in Table 1 and the values of minimum, maximum and energy of the solutions in Table 2.

	domain	initial function
Rectangle	$]0, 2[\times]0, 1[$	$20 \cos(\pi x) \cos(\pi y) x(x-2)y(y-1)$
Disc	$B(0, 1)$	$-800 \cos\left(\frac{\pi}{2}(x^2 + y^2)^{\frac{1}{2}}\right)$
Square	$] -1, 1[^2$	$20 \sin(\pi x) \sin(\pi y)(x+y-0.4)(x+y-1.6)$ $(x-y-0.6)(x-y+0.6)$

TABLE 1. Equation $-\Delta u = u^3$

	$\min u$	$\max u$	$J(u^+)$	$J(u^-)$
Rectangle	-6.6	6.6	37.2	37.2
Disc	-5.85	5.85	29.7	29.7
Square	-5.2	5.2	23.9	23.9

TABLE 2. Characteristics of the approximate solutions

The computations of the entropies have been performed using Riemann sums with mid-point rule and step $\mu = \frac{2}{10000}$. On the one hand, we have the estimate

$$|\nabla(\cos(\theta)v_1 + \sin(\theta)v_2)| \leq \pi\sqrt{5} \leq 12$$

and the function takes its values in $[-2, 2]$, whereas, on the other hand, it is straightforward that $4(1 + \log 4)$ is a Lipschitz constant for the function $t^2 \log t^2$ on $[-2, 2]$. We therefore infer that the approximation error is bounded by

$$\frac{144}{5000} = 0.0288 \tag{7}$$

while the roundoff error is negligible (of the order 10^{-4}) in front of this.

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