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Elementary Kaluza-Klein Towers revisited

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Abstract

Considering that the momentum squared in the extra dimensions is the physically relevant quantity for the generation of the Kaluza-Klein mass states, we have reanalyzed mathematically the procedure for five dimensional scalar fields within the Arkhani-Ahmed, Dimopoulos and Dvali scenario. We find new sets of physically allowed boundary conditions. Beside the usual results, they lead to new towers with non regular mass spacing, to lonely mass states and to tachyons. We remark that, since the SO(1,4) symmetry is to be broken due to the compactification of the extra dimensions, the speed of light could be different in the fifth dimension. This would lead to the possible appearance of a new universal constant besides $\hbar$ and c.

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1 Introduction

An extra-dimension scenario has been proposed by Arkhani-Ahmed, Dimopoulos and Dvali [1] to try to solve the mass scale hierarchy problem: the Planck mass scale $M_{Pl} \equiv \sqrt{\hbar c/G_N} = 1.22090(9)\ 10^{19}\text{GeV}$ ($\overline{M_{Pl}} = M_{Pl}/\sqrt{8\pi}$) is considerably larger than the weak mass scale $M_{EW} \approx 1\text{Tev}$.

In this scenario, it is usually assumed that the full space is a (4+n)-dimensional space which is taken to be flat and real. The standard model particles are constrained to propagate in our four dimensional world (called a “wall” or a “brane”) whereas gravity only is allowed to propagate in the full (4+n)-dimensional “bulk”. It is then argued that the Planck mass would be related to the (4+n)-dimensional fundamental scale M_* by the relation

$$\overline{M_{Pl}}^2 \approx V_n M_*^{n+2} \ , \tag{1}$$

where V_n is the n-dimensional volume of the n extra dimensions. Assuming equal size $2\pi R$ for each of them, then

$$V_n = (2\pi R)^n \ . \tag{2}$$

The speed of light c and $\hbar$ are used as the only two basic constants of the theory and M_* is chosen of the order of M_{EW}. It is usually concluded that $n = 1$ is excluded. The size $2\pi R$ of the extradimension would then be of the order of 10^8m and much too large as Newton's law is known to hold very well for the sun planets. For $n = 2$, the order of magnitude of $2\pi R$ becomes much smaller, about 100μm, close to where Newton's law has been tested [2] with a sufficient precision. It has been argued that the ADD scenario provides experimental tests. They come through the prediction of the so called Kaluza-Klein mass towers related to the massless graviton.

Similarly an alternative scenario has been proposed by Randall and Sundrum [3] in which the mass scale hierarchy results from an exponential warp factor in a five dimensional non factorizable geometry. The idea that the Standard Model gauge and matter fields propagate in the bulk has also been considered [4].

Experimental searches for extra dimensions have been carried on at the CERN LEP [5], at the TEVATRON [6] and at HERA [7]. They conclude that no statistically significant extra dimension effects have been observed. Further investigations are expected to be persued at the LHC [8].

Here we follow the ADD scenario in the five dimensional case ($n = 1$). We call s the extra dimension supposed to be space like in order to have only one time. Indeed, considerations of two times lead to many difficulties, the ensuing possible breaking of causality being an important one [10]. We restrict ourselves to scalar particles, originating from a real five dimensional massless scalar field assumed to propagate in the bulk. The associated Kaluza-Klein masses m_n are deduced from a natural ansätz expressed by the equation

$$\partial_s^2 \phi^{[n]} n = -m_n^2 \phi^{[n]} \ . \tag{3}$$

The mass eigenvalues m_n^2 are essentially determined by boundary conditions imposed on the scalar fields. As summarized for example by Rizzo [9], periodic boundary conditions where the scalar field moves around a circle in s of radius $2\pi R$, box type conditions where the scalar particle is confined to a strip of s of finite size or other boundary conditions, can be imposed. They lead to the specific physical spacing of the masses of the fields present in the tower. In particular, they govern the presence or absence of a zero mass field at the bottom of the tower.

In this article, starting from basic physical considerations, we study the minimal set of coherent physical requirements which should lead to the boundary conditions. We transpose these physical ideas in a precise mathematical sense and discuss all the ensuing allowed boundary conditions and the resulting physical Kaluza-Klein states.

The article is organised as follows. In section (2), we present and discuss the physical ideas. From this discussion, all the allowed boundary conditions are deduced. In section (3), we construct the scalar mass states which follow from the boundary conditions applied to a real massless scalar field in five dimensions. Some towers are new (3.1) and lonely KK states are expected (3.2). For completeness, we show that tachyon states may exist (3.3) for certain specific boundary conditions. In section (3.4), we show explicitly on one example how one of the new towers develops. Finally, in section (4), we stress our new findings and outline the interest of the extension of our approach to particles of higher spin.

2 Physical considerations and boundary conditions

As stated above, we limit ourselves to a discussion involving one extra dimension labeled by s and supposed to be space like. It is assumed that, though in our space the variables x^μ ($\mu = 0, \ldots, 3$) appear to span the full $[-\infty, +\infty]$ range, the extra dimension s is of finite extension $[0, 2\pi R]$. It should be remarked immediately that this hypothesis destroys the generalized Minkowski $\mathrm{SO}(1,4)$ invariance which could be potentially present. The minimal symmetry is therefore restricted to SO(1,3).

Essentially one can then imagine three major types of geometries. The space is like a ribon and our brane is at one edge in the extra dimension, say at the point $s = 0$. The space is like a cylinder and the point $\{s = 0\}$ is identical to the point $\{s = 2\pi R\}$. Our brane is located say at that point $s = 0 = 2\pi R$ and is a generator of the cylinder. The space is like a ribon and our brane is not at an edge but somewhere inside, say at the point located at some point $\hat{s}$, $0 < \hat{s} < 2\pi R$ within the allowed s range.

Though the SO(1,4) symmetry is broken from the start, it is usually assumed that s and x^i ($i = 1, 2, 3$) are commensurable. To study in a precise way a minimal deviation from this symmetry, let us first define the length squared in the five dimensional space as by

$$L^2 = (ct)^2 - \sum_{i=1}^{3} (x^i)^2 - a^2 s^2 \ . \tag{4}$$

It is obvious, using the variables to $t' = ct$, $x'^i = x^i$, $s' = as$, that this distance is fully SO(1,4) invariant. The natural invariant equation for a massless scalar field is then

$$\left(\frac{1}{c^2} \partial_t^2 - \sum_{i=1}^{3} \partial_{x_i}^2 - \frac{1}{a^2} \partial_s^2 \right) \Phi(x_\mu, s) = 0 \tag{5}$$

and the speed of light along the s direction is equal to the speed of light in the x^i directions.

A minimal deviation from the SO(1,4) symmetry would be to replace the parameter a in (5) by an independent parameter b in the form

$$\left(\frac{1}{c^2} \partial_t^2 - \sum_{i=1}^{3} \partial_{x_i}^2 - \frac{1}{b^2} \partial_s^2 \right) \Phi(x_\mu, s) = 0 \ . \tag{6}$$

The speed c of a massless particle in the x^i direction is different from the speed ca/b in the s direction. The parameter b/a, a priori unknown and unpredictable, is a measure of the violation of SO(1,4) and of the commensurability of the length units in the x^i and s directions.

To keep things as simple as possible, we have put $a = b = c = 1$. The parameter a/b does not play any role in the mathematics that follows. The possibility $a/b \neq 1$ should be kept in mind once experimental consequences are looked for and masses are evaluated in terms of $1/R$.

We then follows the usual pattern, supposing a separation of variables in the form (where n is some summation index)

$$\Phi(x_\mu, s) = \sum_n \psi^{[n]}(x_\mu)\phi^{[n]}(s) \tag{7}$$

with $\psi^{[n]}(x_\mu)$ and $\phi^{[n]}(s)$ both real by construction.

The key ansätz of the approach is to suppose that $\phi^{[n]}(s)$ satisfies the equation

$$\partial_s^2\phi^{[n]}(s) = -m_n^2\phi^{[n]}(s) \tag{8}$$

with m_n^2 real. One then finds from Eq. (5) (with $a = 1$)

$$(\Box_4 + m_n^2)\psi^{[n]}(x_\mu) = 0\ . \tag{9}$$

If m_n^2 is assumed to be positive (or zero), this corresponds to an ordinary massive (or massless) scalar four dimensional particle, a so-called Kaluza-Klein mass state. If m_n^2 happens to be negative, this would correspond to a tachyon.

On the finite s region, the Eq. (8) is usually solved by imposing, for example, periodic, antiperiodic or box like boundary conditions

$$\begin{array}{ll}
\left\{\begin{array}{rcl} \phi^{[n]}(2\pi R) &=& \phi^{[n]}(0) \\ \partial_s\phi^{[n]}(2\pi R) &=& \partial_s\phi^{[n]}(0) \end{array}\right\} & \text{Periodic Boundary Conditions ,} \\
\left\{\begin{array}{rcl} \phi^{[n]}(2\pi R) &=& -\phi^{[n]}(0) \\ \partial_s\phi^{[n]}(2\pi R) &=& -\partial_s\phi^{[n]}(0) \end{array}\right\} & \text{Antiperiodic Boundary Conditions ,} \\
\left\{\begin{array}{rcl} \phi^{[n]}(0) &=& 0 \\ \phi^{[n]}(2\pi R) &=& 0 \end{array}\right\} & \text{Box Conditions .}
\end{array} \tag{10}$$

All these conditions follow essentially from the requirement that the operator $P^s = i\partial_s$ as well as its square P_s^2 are symmetric as it is usually assumed for the energy operator and momentum operators P^μ in our four dimensional space.

One may wonder if this is really the most general physical requirement. Indeed, in view of the fact that the physically measurable quantities in our four dimensional subspace are the m_n^2 arising in Eq. (9), which hence have to be real, we conclude that the minimal requirement is to have P_s^2 symmetric and not necessarily P_s itself.

It should be stressed that if on an infinite dimensional real space ($[-\infty, \infty]$ range) the notion of symmetric and selfadjoint momentum operators are essentially identical, this is not automatically the case on a space with finite extent [11].

Our physical hypothesis translates then into the mathematically more precise hypothesis

$$\Big\{\text{Basic Hypothesis :} \quad P_s^2 \text{ is a symmetric operator}\Big\} \ . \tag{11}$$

Let $\phi_a(s)$ and $\phi_b(s)$ be two vectors in the relevant Hilbert space with scalar product (remember that we have restricted ourselves to real fields)

$$(\phi_a, \phi_b) \equiv \int_0^{2\pi R} \phi_a(s)\phi_b(s) \ ds \ . \tag{12}$$

The basic hypothesis (11) is equivalent to the statement

$$(\phi_a, P_s^2\phi_b) = (P_s^2\phi_a, \phi_b) \tag{13}$$

which leads to the conditions

$$\begin{aligned} &\Big[(\partial_s\phi_a)\,\phi_b\Big]_{s=2\pi R} - \Big[(\partial_s\phi_a)\,\phi_b\Big]_{s=0} \\ &= \Big[\phi_a\,(\partial_s\phi_b)\Big]_{s=2\pi R} - \Big[\phi_a\,(\partial_s\phi_b)\Big]_{s=0} \quad . \end{aligned} \tag{14}$$

From this equation, all the specific boundary conditions which guarantee that P_s^2 is symmetric can easily be deduced. Boundary conditions must relate linearly and homogeneously the four quantities (notation $\partial_s\phi(a) \equiv \partial_s\phi|_{s=a}$)

$$\phi(0), \quad \phi(2\pi R), \quad \partial_s\phi(0), \quad \partial_s\phi(2\pi R) \ . \tag{15}$$

In the Hilbert space of square integrable functions on $[0, 2\pi R]$, they constrain the vectors to be in the domain of a symmetric P_s^2 operator. Hence the same relations have to hold for ϕ_a and ϕ_b independently.

It is convenient to build all possible sets of boundary conditions and classify them canonically according to the number of independent homogeneous linear relations they are supposed to satisfy. The general discussion leads to numerous cases with four relations (Case indexed by A), with three relations (Cases indexed by B) and with two relations (Cases indexed by C). Eq. (14) cannot be satisfied if there is one relation only.

The results are summarized in Table (1). The λ_i $(i = 1, 2, 3)$, μ_i $(i = 1, 2)$, α_i $(i = 1, \dots, 4)$, ρ_i $(i = 1, 2)$, ν, κ and ζ are arbitrary real parameters which specify the relevant domains of the corresponding operators P_s^2.

3 Kaluza-Klein mass states

In this section we consider and solve the basic equation (8) which determines the masses of the Kaluza-Klein states. There are mass towers, lonely states and tachyons. Remind that all the boundary conditions of Table (1) derive from the imposition that the operator P_s^2 is symmetric and hence that its eigenvalues m_n^2 are real.

3.1 Towers

Let us first suppose that m_n^2 is positive corresponding to ordinary scalar particles. Eq. (8) is trivial to solve irrespective of the boundary conditions

$$\phi_n(x^\mu, s) = \sigma_n \sin(m_n s) + \tau_n \cos(m_n s) \tag{16}$$

where σ_n and τ_n are real functions of x^μ. In fact, for the same mass squared value there exists two independent eigenvectors where the variables separate. The derivative writes trivially

$$\partial_s \phi_n(s) = m_n \sigma_n \cos(m_n s) - m_n \tau_n \sin(m_n s)\ . \tag{17}$$

We introduce the notation

$$\begin{aligned} S_n &= \sin(2\pi m_n R) \\ C_n &= \cos(2\pi m_n R) \\ S_n^2 + C_n^2 &= 1 \end{aligned} \tag{18}$$

and, dropping the index n for the time being, write

$$\begin{aligned}
\phi(0) &= \tau && (19)\\
\partial_s\phi(0) &= m\sigma && (20)\\
\phi(2\pi R) &= \sigma S + \tau C && (21)\\
\partial_s\phi(2\pi R) &= m\sigma C - m\tau S\ . && (22)
\end{aligned}$$

It is then a matter of simple algebra to replace these quantities successively in the different allowed boundary relations enumerated in Table (1). Each of the cases obviously leads to restrictions for the values of the functions τ and σ and the parameter m of the solutions and relates them to the size $2\pi R$ of the s-space. In a few cases, the boundary conditions are inconsistent with the form of the solution. In other cases, the compatibility of the equations restricts the parameters of the allowed boundary conditions. In particular, four boundary relations (Case A of Table (1)) do not lead to any non trivial solution. Many types of situations occur. In general, an infinite set of masses constituting the Kaluza-Klein towers appears. For boundary conditions involving three relations they are summarized in Table (2) and for two relations in Table (3). In almost all of these cases there is no field of zero mass at the bottom of the tower. It should be remarked that in the classical Kaluza-Klein towers, i.e. when P_s is selfadjoint, the spacing between the masses is regular. Here, as will be seen in an example in section (3.4), in many cases deviations from the regular spacing occur. In a small set of cases, the infinite tower has a mass zero state at its bottom. We have grouped these cases together in Table (4).

3.2 Lonely masses

In a few cases, a lonely state appears when very specific relations are satisfied between the parameters of the boundary conditions and the size $2\pi R$ of the s region. These lonely states are summarized in Table (5).

3.3 Tachyons

In the preceding subsections we have solved the basic equation (8) supposing that the mass squared eigenvalues of the symmetric operator of P_s^2 are positive or zero. We here study the case of m_n^2 negative and write $h_n^2 = -m_n^2$. The solutions, when they exist, will correspond to tachyons arising for certain types of boundary conditions. Eq. (8) is again trivial to solve irrespective of the boundary conditions. The real solutions (dropping the n index for simplicity) are

$$\phi(s) = \widetilde{\sigma}\, e^{hs} + \widetilde{\tau}\, e^{-hs} \tag{23}$$

where $\widetilde{\sigma}$ and $\widetilde{\tau}$ are real.

Going through all the sets of boundary conditions, one finds in a certain number of cases lonely tachyons where the h^2 as well as the size $2\pi R$ of the s region are fixed in terms of the parameters. They are summarized in Table (6) for the cases corresponding to three boundary relations, and in Table (7) for two boundary relations.

3.4 Example

As an explicit example of the mass structure of the new Kaluza-Klein towers, we consider the Case B1b-tw (see Table (2)). The masses m_n are the solutions of the equation

$$\begin{aligned} \tan(2\pi m_n R) &= \frac{2 m_n \lambda}{m_n^2 - \lambda^2} \\ \mathrm{sign}(\sin(2\pi m_n R)) &= \epsilon\ \mathrm{sign}(\lambda), \quad m_n > 0,\ \lambda \neq 0 \end{aligned} \tag{24}$$

where λ is a real arbitrary parameter and ϵ is the sign which appears in the boundary condition corresponding to the Case B1b-tw (Table (2)). By renormalisation with $m_n = \overline{m}_n/R$ and $\lambda = \overline{\lambda}/R$, R may be set equal to 1. The solutions are found at the intersections of the two curves representing the left and right hand sides of (24) as functions of $\overline{m}_n$ for a given value of $\overline{\lambda}$. This is shown for $\overline{\lambda} = 2.63$ in Figure (4). In Table (8), the successive masses are given explicitly for various values of $\overline{\lambda}$.

Let us comment these results.

For $\overline{\lambda} = \infty$, the spacing is regular ($\tan(2\pi\overline{m}_n) = 0 \mapsto \overline{m}_n = n/2$) which defines n. The masses for n even and n odd belong to the towers $\epsilon = -1$ and $\epsilon = 1$ respectively. When $\overline{\lambda}$ decreases, deviations from the regularity appear. In fact $\overline{m}_n$ decreases continuously with $\overline{\lambda}$. When one reaches $\overline{\lambda} = -\infty$ (jumping over $\overline{\lambda} = 0$ which is in principle forbidden), one finds again the regularity but with n displaced by two units.

It is useful to study Eq. (24) for very small values of m_n. The equation reduces to (up to a factor m_n)

$$\overline{m}_n^2 = \overline{\lambda}\left(\overline{\lambda} + \frac{1}{\pi}\right) \ . \tag{25}$$

We see that $\overline{m}_n^2$ may become very small in two cases: when $\overline{\lambda} \to 0$ and when $\overline{\lambda} \to -1/\pi$. This justifies two properties of Table (8). First, looking at the column indexed by $n = 1$, one sees that $\overline{m}_1$ goes to zero when $\overline{\lambda}$ decreases to zero with no states existing anymore for $\overline{\lambda}$ negative. Then, looking at column $n = 2$, one sees that $\overline{m}_2$ goes to zero when $\overline{\lambda}$ decreases to $\overline{\lambda} = -1/\pi \approx -0.31831$ with again no states existing anymore for $\overline{\lambda} < -1/\pi$.

When $\overline{\lambda}$ is of the form of $p + 1/4$ or $p + 3/4$ where p is an integer, there exists a mass in the tower which is exactly equal to $\overline{m}_n = |\overline{\lambda}|$ (for example see $\overline{\lambda} = \{0.25, -0.25, -0.75\}$ in the Table).

4 Conclusions

Coming back critically to the elementary Kaluza-Klein towers, we have studied in detail the case of a real five dimensional scalar field assumed to propagate in the bulk.

The key determination of the KK mass squared states results from the diagonalisation of the operator P_s^2. The physically observable m^2 are real and hence P_s^2 must be a symmetric operator. Analyzing mathematically this restriction, we have considered all the allowed relevant boundary conditions. For many of them, P_s is not symmetric. One should remark that the "box" boundary conditions ($\phi(0) = 0$, $\phi(2\pi R) = 0$, see Case C5 in (1)) appear in the set. Moreover the "periodic" or "antiperiodic" boundary conditions appear as particular cases of more general boundary conditions (see Case C1 in Table (1)).

All the corresponding KK states have then been deduced from our general boundary conditions,

Summarizing our results, we found that, in general, the KK towers consist of regular (equally spaced) and quite often of non regular sequences of states with in a few cases a zero mass state at the bottom. We found that for some specific values of the parameters fixing the boundary conditions, some lonely KK states may exist. For completenes, we also looked at the possiblity of tachyon KK states ($m^2 < 0$) and found that they would appear as isolated states. For a massive (M) five dimensional scalar field, all the KK mass squared states are simply shifted by M^2.

We have seen that our new boundary conditions for scalar fields lead to original physical consequences. Extensions of our approach to higher dimensional spaces should be considered. Along the same line, a careful investigation, a priori non trivial, of the physical conditions on boundary conditions for higher spins should be carried on. In particular spin one and spin two deserve special studies.

In our work, essentially no dynamical consideration have been made as to the possibility of finding experimental evidence for the existence of the potentially predicted KK mass states. Considering that the SO(1,4) invariance is anyhow due to be broken, a word of caution has been put forward about the generally implicitly accepted view that there is just the same usual speed of light on the brane and in the bulk and hence a unique relation between energy and length.

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Table 1: Table of Allowed Boundary Conditions for P_s^2 symmetric

Boundary Conditions [Symmetric $(P^s)^2$]	
Case	Boundary Conditions
Four boundary relations	
A	$\phi(0) = 0$ $\phi(2\pi R) = 0$ $\partial_s\phi(0) = 0$ $\partial_s\phi(2\pi R) = 0$
Three boundary relations	
B1	$\phi(2\pi R) = \lambda_1\phi(0)$ $\partial_s\phi(0) = \lambda_2\phi(0)$ $\partial_s\phi(2\pi R) = \lambda_3\phi(0)$
B2	$\phi(0) = 0$ $\phi(2\pi R) = \mu_1\partial_s\phi(0)$ $\partial_s\phi(2\pi R) = \mu_2\partial_s\phi(0)$
B3	$\phi(0) = 0$ $\partial_s\phi(0) = 0$ $\partial_s\phi(2\pi R) = \nu\phi(2\pi R)$
B4	$\phi(0) = 0$ $\phi(2\pi R) = 0$ $\partial_s\phi(0) = 0$
Two boundary relations	
C1	$\phi(2\pi R) = \alpha_1\phi(0) + \alpha_2\partial_s\phi(0)$ $\partial_s\phi(2\pi R) = \alpha_3\phi(0) + \alpha_4\partial_s\phi(0)$ $\alpha_1\alpha_4 - \alpha_2\alpha_3 = 1$
C2	$\partial_s\phi(0) = \rho_1\phi(0)$ $\partial_s\phi(2\pi R) = \rho_2\phi(2\pi R)$
C3	$\phi(0) = 0$ $\partial_s\phi(2\pi R) = \kappa\phi(2\pi R)$
C4	$\phi(2\pi R) = 0$ $\partial_s\phi(0) = \zeta\phi(0)$
C5	$\phi(0) = 0$ $\phi(2\pi R) = 0$

Table 2: Table of Kaluza-Klein towers for three boundary relations

Boundary Conditions and Towers for a Real Scalar Field $m_n > 0$ Three boundary relations		
Case	Boundaries $(\epsilon^2 = 1)$	Type of Tower
B1a-tw	$\phi(2\pi R) = \epsilon\phi(0)$ $\partial_s\phi(0) = \lambda\phi(0) \qquad (\lambda \neq 0)$ $\partial_s\phi(2\pi R) = \epsilon\lambda\phi(0)$	$\cos(2\pi m_n R) = \epsilon \;\; \left(\tau_n = \frac{m_n}{\lambda}\sigma_n\right)$ $\mapsto m_n = \frac{n}{2R}$ $\epsilon = \;\; 1 \to n$ even $\epsilon = -1 \to n$ odd
B1b-tw	$\phi(2\pi R) = \epsilon\phi(0)$ $\partial_s\phi(0) = \lambda\phi(0) \qquad (\lambda \neq 0)$ $\partial_s\phi(2\pi R) = -\epsilon\lambda\phi(0)$	$\tan(2\pi m_n R) = \frac{2m_n\lambda}{m_n^2 - \lambda^2} \;\; \left(\tau_n = \frac{m_n}{\lambda}\sigma_n\right)$ if $\mathrm{sign}(\sin(2\pi m_n R)) = \epsilon \;\mathrm{sign}(\lambda)$ $(m_n \neq 0)$
B1c-tw	$\phi(2\pi R) = \epsilon\phi(0)$ $\partial_s\phi(0) = 0$ $\partial_s\phi(2\pi R) = 0$	$\cos(2\pi m_n R) = \epsilon \;\; (\sigma_n = 0)$ $\mapsto m_n = \frac{n}{2R}$ $\epsilon = \;\; 1 \to n$ even $\epsilon = -1 \to n$ odd
B2-tw	$\phi(0) = 0$ $\phi(2\pi R) = 0$ $\partial_s\phi(2\pi R) = \epsilon\partial_s\phi(0)$	$\cos(2\pi m_n R) = \epsilon \;\; (\tau_n = 0)$ $\mapsto m_n = \frac{n}{2R}$ $\epsilon = \;\; 1 \to n$ even $\epsilon = -1 \to n$ odd

Table 3: Table of Kaluza-Klein towers for two boundary relations. In Case C1c-tw, $F(m)$ is a solution of $(\alpha_1^2 m^2 - m^2 + \alpha_3^2)F^2 + 2(\alpha_1\alpha_2 m^2 + \alpha_4\alpha_3)mF + (\alpha_2^2 m^2 + \alpha_4^2 - 1)m^2 = 0$ and $G(m,F) = \frac{-\alpha_3 F^2 + (\alpha_1 - \alpha_4)mF + \alpha_2 m^2}{\alpha_1 m F^2 + (\alpha_2 m^2 + \alpha_3)F + \alpha_4 m}$

Boundary Conditions and Towers for a Real Scalar Field $m_n > 0$ Two boundary relations		
Case	Boundaries $(\epsilon^2 = 1)$	Type of Tower
C1a-tw	$\phi(2\pi R) = \epsilon\phi(0)$ $\partial_s\phi(2\pi R) = \alpha_3\phi(0) + \epsilon\partial_s\phi(0)$	$\cos(2\pi m_n R) = \epsilon \quad (\tau_n = 0)$ $\mapsto m_n = \frac{n}{2R}$ $\epsilon = \ \ 1 \to n$ even $\epsilon = -1 \to n$ odd
C1b-tw	$\phi(2\pi R) = \epsilon\phi(0) + \alpha_2\partial_s\phi(0)$ $\partial_s\phi(2\pi R) = \epsilon\partial_s\phi(0)$	$\cos(2\pi m_n R) = \epsilon \quad (\sigma_n = 0)$ $\mapsto m_n = \frac{n}{2R}$ $\epsilon = \ \ 1 \to n$ even $\epsilon = -1 \to n$ odd
C1c-tw	$\phi(2\pi R) = \alpha_1\phi(0) + \alpha_2\partial_s\phi(0)$ $\partial_s\phi(2\pi R) = \alpha_3\phi(0) + \alpha_4\partial_s\phi(0)$ $\alpha_1\alpha_4 - \alpha_3\alpha_2 = 1$	$\tan(2\pi m_n R) = G(m_n, F(m_n))$ $\tau_n/\sigma_n = F(m_n)$ see caption
C2a-tw	$\partial_s\phi(0) = \rho\phi(0) \qquad (\rho \neq 0)$ $\partial_s\phi(2\pi R) = \rho\phi(2\pi R)$	$\sin(2\pi m_n R) = 0 \quad \left(\tau_n = \frac{m_n}{\rho}\sigma_n\right)$ $\mapsto m_n = \frac{n}{2R}$ n integer
C2b-tw	$\partial_s\phi(0) = 0$ $\partial_s\phi(2\pi R) = 0$	$\sin(2\pi m_n R) = 0 \quad (\sigma_n = 0)$ $\mapsto m_n = \frac{n}{2R}$ n integer
C2c-tw	$\partial_s\phi(0) = \rho_1\phi(0) \qquad (\rho_1 \neq 0)$ $\partial_s\phi(2\pi R) = \rho_2\phi(2\pi R) \quad (\rho_2 \neq \rho_1)$	$\tan(2\pi m_n R) = \frac{m_n(\rho_1 - \rho_2)}{m_n^2 + \rho_1\rho_2} \quad \left(\tau_n = \frac{m_n}{\rho_1}\sigma_n\right)$
C2d-tw	$\partial_s\phi(0) = 0$ $\partial_s\phi(2\pi R) = \rho_2\phi(2\pi R) \quad (\rho_2 \neq 0)$	$\cot(2\pi m_n R) = -\frac{m_n}{\rho_2} \quad (\sigma_n = 0)$
C3a-tw	$\phi(0) = 0$ $\partial_s\phi(2\pi R) = \kappa\phi(2\pi R) \quad (\kappa \neq 0)$	$\tan(2\pi m_n R) = \frac{m_n}{\kappa} \quad (\tau_n = 0)$
C3b-tw	$\phi(0) = 0$ $\partial_s\phi(2\pi R) = 0$	$\cos(2\pi m_n R) = 0 \quad (\tau_n = 0)$ $\mapsto m_n = \frac{n}{2R} + \frac{1}{4R}$ n integer
C4a-tw	$\phi(2\pi R) = 0$ $\partial_s\phi(0) = \lambda\phi(0) \qquad (\lambda \neq 0)$	$\tan(2\pi m_n R) = -\frac{m_n}{\lambda} \quad \left(\tau_n = \frac{m_n}{\lambda}\sigma_n\right)$
C4b-tw	$\phi(2\pi R) = 0$ $\partial_s\phi(0) = 0$	$\cos(2\pi m_n R) = 0 \quad (\sigma_n = 0)$ $\mapsto m_n = \frac{n}{2R} + \frac{1}{4R}$ n integer
C5-tw	$\phi(0) = 0$ $\phi(2\pi R) = 0$	$\sin(2\pi m_n R) = 0 \quad (\tau_n = 0)$ $\mapsto m_n = \frac{n}{2R}$ n integer

Table 4: Table of Kaluza-Klein towers with a zero mass state

Real Scalar Tower with a natural zero mass $m_0 = 0$ at the bottom		
Case	Boundaries	Tower
B1-z	$\phi(2\pi R) = \phi(0)$ $\partial_s\phi(0) = 0$ $\partial_s\phi(2\pi R) = 0$	$\cos(2\pi m_n R) = 1 \quad (\sigma_n = 0)$ $\mapsto m_n = \frac{n}{2R}$ n even
C1-z	$\phi(2\pi R) = \phi(0) + \alpha_2 \partial_s\phi(0)$ $\partial_s\phi(2\pi R) = \partial_s\phi(0)$	$\cos(2\pi m_n R) = 1 \quad (\sigma_n = 0)$ $\mapsto m_n = \frac{n}{2R}$ n even
C2-z	$\partial_s\phi(0) = 0$ $\partial_s\phi(2\pi R) = 0$	$\sin(2\pi m_n R) = 0 \quad (\sigma_n = 0)$ $\mapsto m_n = \frac{n}{2R}$ n integer

Table 5: Table of Lonely Mass Cases

Boundary Conditions, Lonely Mass Cases for a Real Scalar Field $m \neq 0$		
Case	Boundaries	Type of Tower
B1-s	$\phi(2\pi R) = \lambda_1\phi(0)$ $\partial_s\phi(0) = \lambda_2\phi(0) \quad (\lambda_2^2 \neq 1)$ $\partial_s\phi(2\pi R) = \lambda_3\phi(0)$	$m^2 = \frac{\lambda_1^2 - \lambda_3^2}{\lambda_2^2 - 1} \quad \left(\sigma = \frac{\lambda_1}{n}\tau\right)$ if $\tan(2\pi m R) = \frac{m(\lambda_1\lambda_2 - \lambda_3)}{\lambda_1\lambda_3 + \lambda_2 m^2}$
B2-s	$\phi(0) = 0$ $\phi(2\pi R) = \mu_1\partial_s\phi(0) \quad (\mu_1 \neq 0)$ $\partial_s\phi(2\pi R) = \mu_2\partial_s\phi(0)$	$m^2 = \frac{1-\mu_2^2}{\mu_1^2} \quad (\tau = 0)$ if $\tan(2\pi m R) = \frac{m\mu_1}{\mu_2}$
C1a-s	$\phi(2\pi R) = \alpha_1\phi(0) + \alpha_2\partial_s\phi(0)$ $\partial_s\phi(2\pi R) = \alpha_3\phi(0) + \alpha_4\partial_s\phi(0)$ $\alpha_1\alpha_4 - \alpha_3\alpha_2 = 1 \quad (\alpha_2 \neq 0)$	$m^2 = \frac{1-\alpha_4^2}{\alpha_2^2} \quad (\tau = 0)$ if $\tan(2\pi m R) = \frac{m\alpha_2}{\alpha_4}$
C1b-s	$\phi(2\pi R) = \alpha_1\phi(0) + \alpha_2\partial_s\phi(0)$ $\partial_s\phi(2\pi R) = \alpha_3\phi(0) + \alpha_4\partial_s\phi(0)$ $\alpha_1\alpha_4 - \alpha_3\alpha_4 = 1 \quad (\alpha_1^2 \neq 1)$	$m^2 = \frac{\alpha_3^2}{1-\alpha_1^2} \quad (\sigma = 0)$ if $\tan(2\pi m R) = -\frac{\alpha_3}{\alpha_1 m}$
C2-s	$\partial_s\phi(0) = \rho_1\phi(0)$ $\partial_s\phi(2\pi R) = \rho_2\phi(2\pi R)$ $\rho_2 \neq \rho_1,\ \rho_1\rho_2 < 0$	$m^2 = -\rho_1\rho_2 \quad \left(\sigma = \frac{\rho_1}{m}\tau\right)$ if $\sqrt{-\rho_1\rho_2} = \left(\frac{n}{2} + \frac{1}{4}\right)/R$

Table 6: Table of tachyons. Three boundary relations

Tachyons for a Real Scalar Field $m^2 = -h^2 < 0$ Three boundary relations		
Case	Boundaries $(\epsilon^2 = 1)$	Type of Tower
B1a-t	$\phi(2\pi R) = \lambda_2 \phi(0)$ $\partial_s \phi(0) = \lambda_1 \phi(0)$ $\partial_s \phi(2\pi R) = \lambda_1 \lambda_2 \phi(0)$ $\left(e^{2\pi\lambda_1 R} = \lambda_2\right)$	$h = \lambda_1 \quad (\widetilde{\tau} = 0)$
B1b-t	$\phi(2\pi R) = \lambda_2 \phi(0)$ $\partial_s \phi(0) = \lambda_1 \phi(0)$ $\partial_s \phi(2\pi R) = \lambda_1 \lambda_2 \phi(0)$ $\left(e^{-2\pi\lambda_1 R} = \lambda_2\right)$	$h = -\lambda_1 \quad (\widetilde{\sigma} = 0)$
B1c-t	$\phi(2\pi R) = \lambda_2 \phi(0)$ $\partial_s \phi(0) = \lambda_1 \phi(0)$ $\partial_s \phi(2\pi R) = \lambda_3 \phi(0)$ $\left(x = \sqrt{\frac{\lambda_3^2 - \lambda_1^2}{\lambda_2^2 - 1}} > 0,\ e^{2\pi\epsilon x R} = \frac{\epsilon x - \lambda_1}{\epsilon \lambda_2 x - \lambda_3}\right)$	$h = \epsilon x \quad \left(\widetilde{\sigma} = \frac{\epsilon x + \lambda_1}{\epsilon x - \lambda_1}\ \widetilde{\tau}\right)$
B2-t	$\phi(0) = 0$ $\phi(2\pi R) = \mu_1 \partial_s \phi(0)$ $\partial_s \phi(2\pi R) = \mu_2 \partial_s \phi(0)$ $\left(x = \frac{\sqrt{\mu_2^2 - 1}}{\mu_1} > 0,\ e^{2\pi\epsilon x R} = \frac{1}{\mu_2 - x\mu_1}\right)$	$h = \epsilon x \quad (\widetilde{\sigma} = -\widetilde{\tau})$

Table 7: Table of tachyons. Two boundary relations

Tachyons for a Real Scalar Field $m^2 = -h^2 < 0$ Two boundary relations		
Case	Boundaries $(\epsilon^2 = 1)$	Type of Tower
C1a-t	$\phi(2\pi R) = \alpha_1 \phi(0) + \alpha_2 \partial_s \phi(0) \quad (\alpha_2 \neq 0)$ $\partial_s \phi(2\pi R) = \alpha_3 \phi(0) + \alpha_1 \partial_s \phi(0)$ $\left(\alpha_1^2 - \alpha_3 \alpha_2 = 1\right)$ $\left(x = \frac{\sqrt{\alpha_1^2 - 1}}{\alpha_2} > 0,\ e^{2\pi\epsilon x R} = \alpha_1 + \epsilon x \alpha_2\right)$	$h = \epsilon x$
C1b-t	$\phi(2\pi R) = \alpha_1 \phi(0) + \alpha_2 \partial_s \phi(0) \quad (\alpha_2 \neq 0)$ $\partial_s \phi(2\pi R) = \alpha_3 \phi(0) + \alpha_4 \partial_s \phi(0)$ $(\alpha_1 \alpha_4 - \alpha_3 \alpha_2 = 1,\ \alpha_4 \neq \alpha_1)$ $\left(x = \frac{\sqrt{\alpha_1^2 - 1}}{\alpha_2} > 0,\ e^{2\pi\epsilon x R} = \alpha_1 + \epsilon x \alpha_2\right)$	$h = \epsilon x$ $\left(\tilde{\tau} = \left(1 - 2\epsilon \alpha_1 \alpha_2 - 2\alpha_1^2\right) \tilde{\sigma}\right)$
C1ct	$\phi(2\pi R) = \alpha_1 \phi(0) + \alpha_2 \partial_s \phi(0) \quad (\alpha_2 \neq 0)$ $\partial_s \phi(2\pi R) = \alpha_3 \phi(0) + \alpha_4 \partial_s \phi(0)$ $\left(\alpha_1 \alpha_4 - \alpha_3 \alpha_2 = 1,\ (\alpha_1 + \alpha_4)^2 > 4\right)$ $(x$ real root of $\left[\alpha_2^2 x^2 + (\alpha_1 - \alpha_4)\alpha_2 x + (1 - \alpha_1 \alpha_4) = 0\right])$ $\left(e^{2\pi x R} = \alpha_1 + x \alpha_2\right)$	$h = x \ (\tilde{\tau} = 0)$
C1d-t	$\phi(2\pi R) = \alpha_1 \phi(0) + \alpha_2 \partial_s \phi(0) \quad (\alpha_2 \neq 0)$ $\partial_s \phi(2\pi R) = \alpha_3 \phi(0) + \alpha_4 \partial_s \phi(0)$ $(\alpha_1 \alpha_4 - \alpha_3 \alpha_2 = 1)$	$h =$ real root of $\left[\left(\alpha_2^2 h^2 - \alpha_2 (\alpha_1 + \alpha_4) h + (\alpha_1 \alpha_4 - 1)\right) e^{4\pi h R}\right.$ $+4\alpha_2 h e^{2\pi h R}$ $\left. - \left(\alpha_2^2 h^2 + \alpha_2 (\alpha_1 + \alpha_4) h + (\alpha_1 \alpha_4 - 1)\right) = 0\right]$ $\tilde{\sigma} = \frac{(\alpha_2 h - \alpha_1) e^{2\pi h R} + 1}{\left(\alpha_1 + \alpha_2 h - e^{2\pi h R}\right) e^{2\pi h R}} \ \tilde{\tau}$
C1e-t	$\phi(2\pi R) = \alpha_1 \phi(0)$ $\partial_s \phi(2\pi R) = \alpha_3 \phi(0) + \frac{1}{\alpha_1} \partial_s \phi(0)$ $\left(e^{2\pi h R} = \alpha_1\right)$	$h = \frac{\alpha_1 \alpha_3}{\alpha_1^2 - 1} \ (\tilde{\tau} = 0)$
C1f-t	$\phi(2\pi R) = \alpha_1 \phi(0)$ $\partial_s \phi(2\pi R) = \alpha_3 \phi(0) + \frac{1}{\alpha_1} \partial_s \phi(0)$ $\left(e^{2\pi h R} = \alpha_1\right)$	$h =$ real root of $\left[\left(\alpha_1 \alpha_3 - \left(\alpha_1^2 + 1\right) h\right) e^{4\pi h R}\right.$ $+4\alpha_1 h e^{2\pi h R}$ $\left. - \left(\alpha_1 \alpha_3 + \left(\alpha_1^2 + 1\right) h\right) = 0\right]$ $\tilde{\sigma} = \frac{1 - \alpha_1 e^{2\pi h R}}{\left(\alpha_1 - e^{2\pi h R}\right) e^{2\pi h R}} \ \tilde{\tau}$
C2a-t	$\partial_s \phi(0) = \rho \phi(0) \qquad (\rho \neq 0)$ $\partial_s \phi(2\pi R) = \rho \phi(2\pi R)$	$h = \rho \ (\tilde{\rho} = 0)$
C2b-t	$\partial_s \phi(0) = \rho \phi(0) \qquad (\rho \neq 0)$ $\partial_s \phi(2\pi R) = \rho \phi(2\pi R)$	$h = -\rho \ (\tilde{\sigma} = 0)$
C2c-t	$\partial_s \phi(0) = \rho_1 \phi(0)$ $\partial_s \phi(2\pi R) = \rho_2 \phi(2\pi R)$	$h=$ real root of $\left[e^{4\pi h R} = \frac{h^2 - (\rho_1 - \rho_2) h - \rho_1 \rho_2}{h^2 + (\rho_1 - \rho_2) h - \rho_1 \rho_2}\right],\ \left(h^2 \neq \rho_1^2\right)$ $\left(\tilde{\sigma} = \frac{h + \rho_1}{h - \rho_1} \ \tilde{\tau}\right)$
C3-t	$\phi(0) = 0$ $\partial_s \phi(2\pi R) = \kappa \phi(2\pi R) \ (\kappa \neq 0)$	$h=$ real root of $\left[e^{4\pi h R} = \frac{\kappa + h}{\kappa - h}\right],\ \left(h^2 \neq \kappa^2\right)$ $(\tilde{\sigma} = -\tilde{\tau})$
C4-t	$\phi(2\pi R) = 0$ $\partial_s \phi(0) = \lambda \phi(0) \qquad (\lambda \neq 0)$	$h=$ real root of $\left[e^{4\pi h R} = \frac{\lambda - h}{\lambda + h}\right],\ \left(h^2 \neq \lambda^2\right)$ $\left(\tilde{\sigma} = \frac{h + \lambda}{h - \lambda} \ \tilde{\tau}\right)$

Table 8: Case B1b-tw. The $\overline{m}_n$-towers as a function of $\overline{\lambda}$

		n										
		1	2	3	4	5	6	7	8	9	10	11
		ϵ										
		$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$
	∞	0.5	1.	1.5	2.	2.5	3.	3.5	4.	4.5	5	5.5
	100.	0.499	0.997	1.496	1.994	2.493	2.991	3.489	3.988	4.486	4.985	5.483
	10.	0.485	0.97	1.455	1.94	2.425	2.91	3.396	3.883	4.369	4.857	5.344
	1.	0.384	0.788	1.219	1.672	2.14	2.617	3.1	3.587	4.077	4.569	5.063
	0.25	0.25	0.622	1.073	1.551	2.039	2.532	3.027	3.523	4.02	4.518	5.016
	0.1	0.17	0.557	1.031	1.521	2.016	2.513	3.011	3.51	4.008	4.508	5.007
	0.01	0.057	0.507	1.004	1.503	2.002	2.502	3.002	3.501	4.001	4.501	5.001
$\overline{\lambda}$	0.001	0.018	0.5	1.	1.5	2.	2.5	3.	3.5	4.	4.501	5.001
	− 0.001		0.5	1.	1.5	2.	2.5	3.	3.5	4.	4.5	5.
	− 0.01		0.494	0.997	1.498	1.999	2.499	2.999	3.5	4	4.5	5
	− 0.1		0.427	0.968	1.479	1.984	2.488	2.99	3.491	3.993	4.493	4.994
	− 0.25		0.25	0.916	1.446	1.96	2.468	2.974	3.478	3.981	4.483	4.985
	− 0.3183		0.004	0.891	1.431	1.949	2.46	2.966	3.471	3.975	4.478	4.98
	− 0.3184			0.891	1.431	1.949	2.46	2.966	3.471	3.975	4.478	4.98
	− 0.75			0.75	1.338	1.88	2.404	2.92	3.432	3.941	4.447	4.953
	− 1.			0.693	1.291	1.842	2.374	2.895	3.41	3.921	4.43	4.937
	− 5.			0.534	1.067	1.599	2.129	2.656	3.181	3.703	4.224	4.742
	− 20.			0.509	1.017	1.525	2.033	2.541	3.049	3.557	4.064	4.572
	− 100.			0.502	1.004	1.505	2.007	2.508	3.01	3.512	4.013	4.515
	$-\infty$			0.5	1.	1.5	2.	2.5	3.	3.5	4.	4.5

Figure 1: The KK tower masses at the intersection of $\tan(2\pi\overline{m}_n)$ and $2\overline{\lambda}\overline{m}_n/\left(\overline{m}_n^2 - \overline{\lambda}^2\right)$ for $\overline{\lambda} = 2.63$, as functions of $\overline{m}_n$ (Case B1b-tw in Table (2))