

A note about a pure spin-connection formulation of General Relativity and spin-2 duality in (A)dS

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Abstract

We investigate the problem of finding a pure spin-connection formulation of General Relativity with non-vanishing cosmological constant. We first revisit the problem at the linearised level and find that the pure spin-connection, quadratic Lagrangian, takes a form reminiscent to Weyl gravity, given by the square of a Weyl-like tensor. Upon Hodge dualisation, we show that the dual gauge field in (A)dS_D transforms under $GL(D)$ in the same representation as a massive graviton in the flat spacetime of the same dimension. We give a detailed proof that the physical degrees of freedom indeed correspond to a massless graviton propagating around the (anti-) de Sitter background and finally speculate about a possible nonlinear pure-connection theory dual to General Relativity with cosmological constant.

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1 Introduction

The history of pure-connection formulation of General Relativity (GR) is old, starting with works of Eddington and Schrödinger [1, 2] — see *e.g.* Section 2 of [3], for a state of the art of pure-connection formulations of gravity. On the one hand, in the case of Eddington–Schrödinger’s proposal, the connection is torsionless but admits nonmetricity off-shell. The Lagrangian density is proportional to the square-root of the determinant of the Ricci tensor. For dimensional reason, the Lagrangian density must be divided by the cosmological constant (we will always work in the geometric unit system: $c = G = 1$) in four spacetime dimension. Upon identifying the metric with the Ricci tensor of the curvature for the connection divided by the cosmological constant, the field equations state that the connection is metric-compatible. Clearly, this action and identification are only well-defined for a nonvanishing cosmological constant $\Lambda \neq 0$. The on-shell value of Eddington–Schrödinger’s action is equal to the product of Λ with the spacetime volume, as is the case for the Einstein–Hilbert action with cosmological constant. The Palatini action plays the rôle of a first-order parent action for both of these second-order actions. In fact, when $\Lambda \neq 0$ the metric is an auxiliary field and the Eddington–Schrödinger action can be obtained by solving the field equation for the metric in terms of the connection inside the Palatini action.

On the other hand, gravity can also be geometrically formulated à la Cartan, with vielbein (*i.e.* an orthonormal frame) and spin-connection (*i.e.* a metric-compatible connection admitting torsion off-shell) as dynamical variables. For GR with nonvanishing cosmological constant, the Lagrangian density in the Cartan formulation admits⁴ a Yang–Mills-like form, *i.e.* quadratic in the curvature, via the MacDowell–Mansouri action [5] which is also a parent action for gravity. One can in principle integrate out the vielbein so as to reach a final action that would be of pure spin-connection type.⁵ In dimensions $D > 3$, this way of proceeding is involved and, for technical reasons, does not lead to a closed expression for the fully nonlinear Lagrangian density for the spin-connection. However, it can be obtained perturbatively around any solution, for instance: (anti-) de Sitter background, as was done till cubic interaction level in the interesting paper [8].

In this note we do not claim to reach the corresponding explicit form of the fully nonlinear, pure spin-connection formulation of GR with cosmological constant, but believe that the formulation we give at the linearised level is simple and suggestive enough to allow for some progress toward the searched-for action. With this goal in mind, we speculate about a possible parent action for GR that would be viewed on the same footing as the MacDowell–Mansouri action.

⁴In the flat case, see [4].

⁵In dimension 3, see [6, 7].

Our Lagrangian takes a form that makes it immediately suited to the holographic renormalisation of the Einstein–Hilbert action with cosmological constant discussed in [9]. This suggests that the resulting pure spin-connection action behaves well under quantisation. One can also put the action as the integral of a square-root featuring a self-dual two-form, upon adding the Pontryagin topological invariant to the Euclideanised action.

The layout of the paper is as follows. In Section 2 we revisit the problem at the linearised level, integrating out the vielbein from the linearised MacDowell–Mansouri action. This leads to a quadratic Lagrangian for the sole spin-connection. Because we keep the Gauss–Bonnet term in the original MacDowell–Mansouri action, even in $D = 4$ where it is a total derivative, we land on a new, suggestive form for the pure spin-connection quadratic Lagrangian $\mathcal{L}(\omega)$. It turns out to be given by a density very reminiscent to the one describing Weyl’s gravity. In the same section, we show that the dual graviton in the (anti-) de Sitter geometry (A)dS $_D$ transforms in a representation of $GL(D)$ identical to the representation carried by the dual *massive* graviton in flat spacetime of the same dimension D and further explain the nature of the dual graviton in (A)dS $_D$.

The core of the paper, in Section 3, consists in giving a detailed proof that the resulting theory propagates only the degrees of freedom for a massless graviton on (A)dS background. This is not obvious indeed, see e.g. [10] where the counting of degrees of freedom was studied for quadratic-curvature-like Lagrangians featuring both the spin-connection and vierbein. A recent discussion on the problem of counting degrees of freedom for pure spin-connection Lagrangians can be found in [3]. In the same section, we explain in which precise sense the field equations for the pure spin-connection action are equivalent to the zero-torsion condition, upon appropriately identifying the *geometrical* vielbein, function of the spin-connection, around (A)dS.

From the results of the previous section, In Section 4 we speculate about a possible fully non-linear, alternative parent action for GR. The proposed action would then be viewed on the same footing as the MacDowell–Mansouri action, considered as a parent action for General Relativity with cosmological constant. The nonlinear Lagrangian we propose reproduces the quadratic Lagrangian $\mathcal{L}(\omega)$ to lowest order. In the same section, we add to the proposed action the Pontryagin term so as to express the resulting euclideanised action as the square root of the determinant of an antisymmetric, (anti-)selfdual two-form. We conclude the paper with a summary in Section 5.

2 Linearised gravity in the pure spin-connection form

2.1 A brief review of MacDowell–Mansouri gravity

MacDowell–Mansouri’s formulation of gravity [5], extended to any dimensions D in [11], is based on a quadratic action in the curvature 2-form of $\mathfrak{so}(1, D - 1)$ or $\mathfrak{so}(2, D - 1)$, depending on the sign of the cosmological constant, later modified by Stelle and West [12] who gave its $\mathfrak{so}(2, D - 1)$ manifestly covariant version. In its original version, it reads

$$S_{MM}[e, \omega] = \frac{1}{2\lambda^2} \int_{\mathcal{M}_D} \epsilon_{abcd k_1 \dots k_{D-4}} \mathcal{R}^{ab}(e, \omega) \wedge \mathcal{R}^{cd}(e, \omega) \wedge e^{k_1} \dots \wedge e^{k_{D-4}}, \quad (1)$$

The Lorentz-valued components of the (A)dS $_{d+1}$ curvature 2-form reads

$$\mathcal{R}^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} + \sigma \lambda^2 e^a \wedge e^b, \quad (2)$$

with ω^{ab} and e^a the spin connection and vielbein 1-forms, respectively. The inverse of the (A)dS radius squared, λ^2 , is related to the cosmological constant by $\lambda^2 = -\sigma \frac{2\Lambda}{(D-1)(D-2)}$. The parameter $\sigma = +1$ corresponds to AdS $_D$ whereas $\sigma = -1$ corresponds to dS $_D$. In the following, for the sake of definiteness, we will take $\sigma = +1$ with the understanding that the results apply to dS $_D$ upon changing the sign in front of λ^2 . The translation-valued components of the (A)dS $_D$ curvature 2-form coincides with the torsion

$$T^a = de^a + \omega^a{}_b \wedge e^b. \quad (3)$$

The above action (1) is invariant under diffeomorphisms and local Lorentz transformations, under which both the vielbein and spin-connection transform in the usual way.

In $D = 4$, the above Lagrangian density contains a total derivative, related to the Gauss–Bonnet invariant

$$I_4[\omega] = \frac{1}{2} \int_{\mathcal{M}_4} \epsilon_{abcd} R^{ab}(\omega) \wedge R^{cd}(\omega), \quad (4)$$

a topological invariant proportional to the Euler characteristic $\chi(\mathcal{M}_4)$. Explicitly, one has $I_4[\omega] = 16\pi^2 \chi(\mathcal{M}_4)$, where $R^{ab}(\omega) = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$ is the Lorentz curvature 2-form. Therefore, in $D = 4$ one can drop it and obtain the Cartan–Weyl action with cosmological constant

$$S_{CW}[e, \omega] = \int_{\mathcal{M}_4} \left(R^{ab}(\omega) + \frac{1}{2} \lambda^2 e^a e^b \right) e^c e^d \epsilon_{abcd}, \quad (5)$$

where from now on we omit the wedge product between differential forms. The equations of motion derived from $S_{CW}[e, \omega]$ are:

$$\epsilon_{abcd} T^c e^d \approx 0, \quad (6)$$

and

$$\epsilon_{abcd} e^b \mathcal{R}^{cd} \approx 0, \quad (7)$$

where the weak equality symbol \approx will stand for any equality only valid when the equations of motion hold. The first set of field equations puts the torsion to zero, and provided that the components of the vierbein are invertible, allows one to express the spin connection in terms of the vierbein. The remaining field equations then provide vacuum Einstein's equations with cosmological constant Λ :

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} \approx 0, \quad (8)$$

As clearly recalled in [13], gravity in this formulation is made more similar to a gauge theory.⁶ Indeed, the action amounts to a Yang–Mills-like action, in the sense that it is quadratic in the Lorentz-valued part of the (A)dS_D curvature 2-form. However, a crucial difference is that it is not of the Yang–Mills type $\int \text{tr}(F \wedge *F)$ but rather of the form $\int \text{tr}(F \wedge F)$. Another crucial difference with respect to the usual Yang–Mills gauge theories is that the underlying geometry is not described in terms of a principal G -bundle, where G is the isometry group of the maximally-symmetric spacetime, but is a Cartan geometry (see [15, 16] for more details) since, among other things, the vielbein is nondegenerate.

In dimension $D > 4$, the term in the Lagrangian that is quadratic in the Lorentz curvature 2-form R^{ab} is no longer a total derivative, but nevertheless the linearised field equations in $D > 4$ still remain equivalent to the Fierz–Pauli equations that propagate a massless spin-2 field around AdS_D — see e.g. Section 2.2 in [14].

At this stage, one could try and integrate out the vielbein from the second set of field equations (7) and obtain the pure spin-connection action $S_{MM}[e(\omega), \omega]$. This action is perfectly well-defined, although obtaining its explicit form remains an open problem, since solving (7) in terms of the vielbein turns out to be technically very involved. We prefer, instead, to integrate out the vielbein from the linearisation of the action (1), and then seek for nonlinear extensions of the resulting, pure spin-connection quadratic action.

2.2 Pure spin-connection action for linearised gravity around AdS_D

Linearising MacDowell–Mansouri's action (1) around AdS_D gives:

$$S_\lambda[h, \omega] = \frac{1}{2\lambda^2} \int_{AdS_D} \epsilon_{abcd k_1 \dots k_{D-4}} \left(\bar{\nabla} \omega^{ab} + 2\lambda^2 \bar{e}^a h^b \right) \left(\bar{\nabla} \omega^{cd} + 2\lambda^2 \bar{e}^c h^d \right) \bar{e}^{k_1} \dots \bar{e}^{k_{D-4}} \quad (9)$$

with $\bar{\nabla} = d + \bar{\omega}$ the Lorentz covariant derivative of the AdS_D background, \bar{e}^a and $\bar{\omega}^{ab}$ being respectively the background vielbein and spin connection, obeying $\bar{\mathcal{R}}^{ab} = d\bar{\omega}^{ab} + \bar{\omega}^a{}_c \bar{\omega}^{cb} + \lambda^2 \bar{e}^a \bar{e}^b = 0$ and $\bar{T}^a = d\bar{e}^a + \bar{\omega}^a{}_b \bar{e}^b = 0$, and h^a and ω^{ab} their respective fluctuations; *i.e.* $e = \bar{e} + h$ while, with some abuse

⁶Nevertheless, this tantalising similarity calls for several important caveats (see e.g. Appendix A.3 in [14]).

of notation, we use the same symbol ω for the full spin connection and its fluctuation around the background. The linearised action is invariant under the following gauge transformations:

$$\delta_\epsilon h^a = \bar{\nabla} \epsilon^a - \bar{e}_b \epsilon^{ab}, \quad \delta_\epsilon \omega^{ab} = \bar{\nabla} \epsilon^{ab} + 2\lambda^2 \bar{e}^{[a} \epsilon^{b]}, \quad (10)$$

where curved (square) brackets surrounding indices denote (anti)symmetrisation with strength one.

The action (9) leads to the equations of motion:

$$\frac{\delta S_\lambda}{\delta h^d} \approx 0 \quad \Leftrightarrow \quad \epsilon_{abcd} k_1 \dots k_{D-4} \left(\bar{\nabla} \omega^{ab} \bar{e}^c + 2\lambda^2 \bar{e}^a \bar{e}^b h^c \right) \bar{e}^{k_1} \dots \bar{e}^{k_{D-4}} \approx 0, \quad (11)$$

whose solution reads:

$$h_\mu^a \approx -\frac{1}{(D-2)\lambda^2} \left(R_\mu^a - \frac{1}{2(D-1)} R \bar{e}_\mu^a \right), \quad (12)$$

where we defined

$$R_{\mu\nu}{}^{ab} = 2\bar{\nabla}_{[\mu} \omega_{\nu]}{}^{ab}, \quad R_\mu^a = \bar{e}_b^\nu R_{\mu\nu}{}^{ab}, \quad \text{and} \quad R = \bar{e}_a^\mu R_\mu^a, \quad (13)$$

that we will call hereafter respectively Riemann-like tensor, Ricci-like tensor and Ricci-like scalar, because of their formal resemblance with the eponymous tensors and scalar. Using the above expression for h_μ^a , the resulting child action can be written as:

$$\begin{aligned} S_\lambda[\omega] &= -\frac{(D-4)!}{2\lambda^2} \int_{AdS_D} \bar{e} d^D x \left(R_{cd}{}^{ab} R_{ab}{}^{cd} - \frac{4}{D-2} R_b^a R_a^b + \frac{2}{(D-1)(D-2)} R^2 \right), \\ &= -\frac{(D-4)!}{2\lambda^2} \int_{AdS_D} \bar{e} d^D x C_{cd}{}^{ab}(\omega) C_{ab}{}^{cd}(\omega), \end{aligned} \quad (14)$$

featuring the Weyl-like tensor

$$C_{cd}{}^{ab}(\omega) := R_{cd}{}^{ab} - \frac{2}{D-2} (\delta^a_{[c} R_{d]}{}^b - \delta^b_{[c} R_{d]}{}^a) + \frac{2}{(D-1)(D-2)} R \delta^a_{[c} \delta^b_{d]}. \quad (15)$$

Let us stress that, even though (15) is formally the same expression as the usual Weyl tensor, it is instead defined in terms of the Riemann-like tensor, Ricci-like tensor and Ricci-like scalar. The Weyl-like tensor is traceless but not symmetric under the exchange of the two pairs of upper and lower antisymmetric indices. In fact, by noticing that the solution (12) is nothing but $h_\mu^a = -\frac{1}{\lambda^2} P_\mu^a$ for the Schouten-like tensor $P_\mu^a := \frac{1}{(D-2)} \left(R_\mu^a - \frac{1}{2(D-1)} \bar{e}_\mu^a R \right)$, the expression (15) can be expressed more compactly in terms of the following 2-form:

$$C^{ab}(\omega) = R^{ab}(\omega) + 2\lambda^2 \bar{e}^{[a} h^{b]}(\omega). \quad (16)$$

Comparing this expression with the linearisation of the AdS_D curvature (2), we readily see that, on the solution (12) for the vielbein in terms of the spin-connection, the actions (9) and (14) are indeed the same.

The field equations derived from (14) read:

$$\begin{aligned} \frac{\delta S_\lambda[\omega]}{\delta \omega_\mu^{ab}} \approx 0 &\Leftrightarrow \bar{\nabla}_\nu R_{ab}{}^{\mu\nu} + \frac{2}{D-2} (\bar{\nabla}_{[a} R_{b]}{}^\mu - \bar{e}^\mu{}_{[a} \bar{\nabla}_\nu R_{|b]}{}^\nu) + \frac{2}{(D-1)(D-2)} \bar{e}_{[a}{}^\mu \bar{\nabla}_{b]} R \approx 0 \\ &\Leftrightarrow \tilde{C}_{ab}{}^\mu := \frac{1}{D-3} \bar{\nabla}_\nu C_{ab}{}^{\mu\nu} \approx 0 . \end{aligned} \quad (17)$$

The left-hand side of the field equations features the Cotton-like tensor to which we will return in the next section. We will also show in which sense the above field equations can be viewed as the zero-torsion condition for the spin-connection. Due to the gauge symmetries of the action under (10), the left-hand side of the above field equations obey the following Noether identities:

$$\bar{\nabla}_\mu \bar{\nabla}_\nu C_{ab}{}^{\mu\nu} \equiv 0, \quad \bar{\nabla}_\nu C_{ab}{}^{a\nu} \equiv 0 . \quad (18)$$

They are simple consequences of the tracelessness of the Weyl-like tensor.

2.3 Dual graviton in (A)dS_D

Hodge duality for linearised gravity around AdS background was discussed, for example, in [17] for the Hamiltonian formulation, and [18] for the Lagrangian formulation. In this subsection, and more fully in the Section 3, we clarify the nature of the degrees of freedom and the $GL(D)$ symmetry of the dual graviton in AdS_D, from the Lagrangian and manifestly Lorentz-covariant point of view.

As we have just commented above, the child action $S_\lambda[\omega]$ derived in (14) from the parent action (9) inherits the gauge symmetry

$$\delta_\epsilon \omega^{ab} = \bar{\nabla} \epsilon^{ab} + 2\lambda^2 \bar{e}^{[a} \epsilon^{b]} , \quad (19)$$

for the remaining gauge field ω . Again, as long as the cosmological constant is nonvanishing, the second term on the right-hand side above is nonzero, which implies that one can gauge fix the trace of $\omega_\mu^{\alpha\beta}$ to zero. As a consequence, in the gauge where $\omega_\mu^{\mu\beta} \equiv 0$, dualising the pair of upper antisymmetric indices of $\omega_\mu^{\alpha\beta}$ in (A)dS_D gives a dual potential

$$\frac{1}{2} \varepsilon_{\alpha\beta\alpha_1\dots\alpha_{D-2}} \omega_\mu^{\alpha\beta} =: \tilde{\omega}_{\mu|\alpha_1\dots\alpha_{D-2}} = \tilde{\omega}_{\mu|[\alpha_1\dots\alpha_{D-2}]} \quad (20)$$

that transforms in the hook-like, irreducible $GL(D)$ representation characterised by

$$\varepsilon^{\mu\nu\alpha_1\dots\alpha_{D-2}} \tilde{\omega}_{\mu|\alpha_1\dots\alpha_{D-2}} \equiv 0 . \quad (21)$$

For example, in (A)dS₄,

$$\tilde{\omega}_{\mu|\alpha_1\alpha_2} \sim \begin{array}{|c|} \hline \square \\ \hline \end{array} , \quad (22)$$

whereas in (A)dS₅,

$$\tilde{\omega}_{\mu|\alpha_1\alpha_2\alpha_3} \sim \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} , \quad (23)$$

This infinite tower of $\mathfrak{so}(1, D - 1)$ tensors satisfies a first-order differential equation that links each of the tensors irreducibly by the action of the $\mathfrak{so}(2, D - 1)$ translation generators [31]. All these $\mathfrak{so}(1, D - 1)$ tensors correspond to the on-shell Weyl tensor and all its on-shell nontrivial covariant derivatives. At one point of spacetime, the data of these tensors is equivalent to all the nontrivial Taylor coefficients of the gravitational field at this point, thereby allowing to reconstruct the field everywhere in an open patch. In other words, these Lorentz tensors are mapped one-to-one to the coefficients of the metric in the normal Riemann coordinates expansion.

In order to prove that our pure spin-connection formulation of linearised gravity with cosmological constant correctly describes the propagating massless spin-2 field, we thus have to show that the only gauge-invariant tensors that are not constrained by the EOM (17) correspond to the various projections of the covariant derivatives of the Riemann-like tensor $R_{ab|cd}$ on the symmetries of the $\mathfrak{so}(1, D - 1)$ tensors displayed in \mathcal{T} . In order to do so, we start by comparing the Lorentz projections of the Riemann-like tensor, together with all its AdS_D covariant derivatives, with the corresponding projections of the derivatives of the EOM in order to eliminate those components that vanish by virtue of (17).

The outcome of this analysis will be that, indeed, the only Lorentz-irreducible projections of the successive covariant derivatives of the Riemann-like tensor that are (i) gauge-invariant and (ii) nonvanishing on the solutions of (17), are in one-to-one correspondence with the Lorentz tensors in the set \mathcal{T} . Furthermore, we will show that the first tensor in this set, that we will call the *primary Weyl tensor* following [22], obeys the D'Alembert equation in AdS_D ,

$$\left(\square - 2\lambda^2(D - 1)\right)W_{abcd} = 0, \quad (27)$$

which, together with the relations linking all the higher Lorentz tensors in \mathcal{T} , ensures the isomorphism of the $\mathfrak{so}(2, D - 1)$ module \mathcal{T} with the unitary irreducible representation of $\mathfrak{so}(2, D - 1)$ specifying the massless graviton, as explained in details in [31] — see also [22] for a review of linearised unfolded systems around maximally-symmetric backgrounds.

3.2 Gauge-invariant and traceless projections of $R_{ab|cd}$

Clearly, upon inspection of (10), there is no gauge-invariant quantity built out of the undifferentiated spin-connection. At first order in the derivatives of ω^{ab} , we decompose $\bar{\nabla}_\mu \omega_\nu^{ab} = \bar{\nabla}_{(\mu} \omega_{\nu)}^{ab} + \bar{\nabla}_{[\mu} \omega_{\nu]}^{ab}$. The first piece transforms with the second symmetrised derivative of parameter ϵ^{ab} and is not invariant. We therefore start the analysis with the second piece $\bar{\nabla}_{[\mu} \omega_{\nu]}^{ab}$ which is, up to an inessential factor of 2, the Riemann-like tensor, and then consider all its symmetrised covariant

derivatives. Recalling the expression for the Schouten-like tensor

$$P_{a|b} = \frac{1}{(D-2)} \left(R_{a|b} - \frac{1}{2(D-1)} \eta_{ab} R \right), \quad (28)$$

together with the decomposition given in (15):

$$R_{cd|}{}^{ab} = C_{cd|}{}^{ab} + 4 \delta^{[a}{}_{[c} P_{d]}{}^{b]}, \quad (29)$$

we see that considering all the symmetrised covariant derivatives of $R_{cd|ab}$ is equivalent to considering separately all the symmetrised covariant derivatives of $C_{cd|ab}$ and of $P_{a|b}$. Notice that we use a vertical bar to separate groups of antisymmetric indices and that the background vielbein has been used to transform all base indices (Greek) into fiber ones (Latin). Since the transformation law of the Riemann-like tensor under (10) is

$$\delta_\epsilon R_{ab|cd} = -2\lambda^2 (\eta_{c[a} \bar{\nabla}_{b]} \epsilon_d - \eta_{d[a} \bar{\nabla}_{b]} \epsilon_c + \eta_{c[a} \epsilon_{b]d} - \eta_{d[a} \epsilon_{b]c}), \quad (30)$$

we see that all its traceless projections are gauge invariant, hence $C_{ab|cd}$ is gauge invariant. As for the transformations of the Ricci-like tensor and Ricci-like scalar, we have

$$\begin{aligned} \delta_\epsilon R_{ab} &= -\lambda^2 \eta_{ab} \bar{\nabla}_c \epsilon^c - \lambda^2 (D-2) (\bar{\nabla}_a \epsilon_b + \epsilon_{ab}), \\ \delta_\epsilon R &= -2\lambda^2 (D-1) \bar{\nabla}_a \epsilon^a, \end{aligned} \quad (31)$$

leading to the transformation law for the Schouten-like tensor:

$$\delta_\epsilon P_{a|b} = -\lambda^2 (\bar{\nabla}_a \epsilon_b + \epsilon_{ab}). \quad (32)$$

As none of the tensors introduced so far are irreducible under the Lorentz group, we proceed now to the Lorentz-irreducible decompositions of the undifferentiated Riemann-like tensor. We use $\mathfrak{so}(1, D-1)$ Young diagrams to specify the various irreducible pieces. The list of these components is given in Table 1. The first three $\mathfrak{so}(1, D-1)$ -irreducible components of the Riemann-like tensor $R_{ab|cd}$ appearing in the table, denoted by W , \hat{J} and \hat{K} , are gauge invariant. They are the 3 irreducible components of the Weyl-like tensor $C_{ab|cd}$. We show in the next subsection that, from these 3 gauge-invariant tensors, only the first piece, W , is not zero on the EOM (17).

3.3 Projections of the first derivative of the equations of motion

The left-hand side of the field equations (17) are gauge invariant. Since they start with the first derivative of the Riemann-like tensor, one could believe that all the gauge-invariant components of the undifferentiated Riemann-like tensor are on-shell nontrivial observables, hence should be part of the set \mathcal{T} defining the $\mathfrak{so}(2, D-1)$ module carrying the physical degrees of freedom. However, since the covariant derivatives in AdS_D do not commute to zero, there can be differential consequences of

Young diagrams	Corresponding tensors
	$W_{ab cd} := I_{ab cd} - \frac{2}{D-2} (\bar{g}_{c[a}I_{b] d} - \bar{g}_{d[a}I_{b] c}) + \frac{2}{(D-1)(D-2)} (\bar{g}_{c[a}\bar{g}_{b]d} - \bar{g}_{d[a}\bar{g}_{b]c}) I,$ with $I_{ab cd} := \frac{1}{6} (R_{ab cd} + R_{cd ab} - R_{c[a b]d} + R_{d[a b]c})$, $I_{bd} := \frac{1}{2} R_{(b d)}$ and $I := \frac{1}{2} R$
	$\hat{J}_{abc d} := J_{abc d} - \frac{1}{D-2} (2\bar{g}_{c[a}J_{b] d} + \bar{g}_{cd}J_{a b}),$ with $J_{abc d} := \frac{1}{2} (R_{[ab c]d} - R_{d[a b]c})$ and $J_{b d} := \frac{2}{3} R_{[b d]}$
	$\hat{K}_{abcd} = R_{[ab cd]}$
	$R_{[a b]}$
	$R_{(a b)} - \frac{1}{D} \bar{g}_{ab}R$
	R

Table 1: Lorentz-irreducible decomposition of the undifferentiated Riemann-like tensor

(17) that lower the derivative order by 2 units, thereby bringing in gauge-invariant components of the undifferentiated Riemann-like tensor. We show that this is indeed the case, and that from the decomposition of the previous subsection, only the traceless tensor $W_{ab|cd}$ survives on-shell and hence enters the set \mathcal{F} at zeroth order in the covariant derivatives of the Riemann-like tensor. The component $W_{ab|cd}$ is called the *primary Weyl tensor*, see [22].

The way to bring down by two units the number of derivatives acting on the spin-connection in the field equation is by computing $\bar{\nabla}_{[d|}\bar{\nabla}_e C_{|ab]|c}{}^e$ and decomposing it under $\mathfrak{so}(1, D-1)$. We find

$$0 \approx \bar{\nabla}_{[d|}\bar{\nabla}_e C_{|ab]|c}{}^e = -\lambda^2(D-3) \left(R_{[ab|d]c} + \frac{2}{D-2} \bar{g}_{c[a}R_{bd]} \right). \quad (33)$$

By virtue of the first Noether identity (18), the above quantity is identically traceless and implies that both $\hat{J}_{abc|d}$ and \hat{K}_{abcd} vanish on-shell. On the other hand, taking the projection of $\bar{\nabla}_d \bar{\nabla}_e C_{ab|c}{}^e$ on the symmetries of the primary Weyl tensor gives

$$\mathbb{P}^{\boxplus}(\bar{\nabla}_d \bar{\nabla}_e C_{ab|c}{}^e) = \bar{\nabla}_e \mathbb{P}^{\boxplus}(\bar{\nabla}_d C_{ab|c}{}^e) + \lambda^2(D-2)W_{ab|cd}. \quad (34)$$

Since the projection $\mathbb{P}^{\boxplus}(\bar{\nabla}_d C_{ab|c}{}^e)$ does not produce any commutator of covariant derivatives acting on the spin-connection, from the various gauge-invariant components of $R_{ab|cd}$, only $W_{ab|cd}$ survives on-shell, as announced.

3.4 Irreducible components of $\bar{\nabla}_e R_{ab|cd}$

No covariant derivatives $\bar{\nabla}_{(a_1} \dots \bar{\nabla}_{a_k)} R$ are gauge invariant, see (31). Similarly, the symmetrised derivatives of the Ricci-like tensor

$$\bar{\nabla}_{(a_1} \dots \bar{\nabla}_{a_k} R_{a_{k+1} a_{k+2})} \quad (35)$$

are not gauge-invariant. Denoting $A_{ab} := R_{[a|b]}$, it is also simple to see that no derivatives

$$\bar{\nabla}_{(a_1} \dots \bar{\nabla}_{a_k} A_{a)b} \quad (36)$$

can be completed into a gauge invariant quantity either. On the other hand, though the Ricci-like tensor is not gauge invariant, appropriate projections of its covariant derivatives can be. Again, instead of using $R_{a|b}$ and R , it is better to consider the Schouten-like tensor $P_{a|b}$ instead. Introducing the following tensor

$$C_{ab|c} := 2\bar{\nabla}_{[b} P_{a]c} , \quad (37)$$

and recalling (32), we find

$$\delta_\epsilon C_{ab|c} = 2\lambda^2 (\bar{\nabla}_{[a} \epsilon_{b]c} - \lambda^2 \eta_{c[a} \epsilon_{b]}) , \quad (38)$$

so that the following tensor is gauge invariant:

$$\tilde{C}_{ab|c} := 2\bar{\nabla}_{[b} P_{a]c} - 2\lambda^2 \omega_{[a|b]c} \equiv C_{ab|c} - 2\lambda^2 \omega_{[a|b]c} . \quad (39)$$

It is however zero on-shell, as we anticipated with our notation in (17). Indeed, using

$$\bar{\nabla}_{[a} R_{bc]|de} = 2\lambda^2 (\eta_{d[a} \omega_{b|c]e} - \eta_{e[a} \omega_{b|c]d}) , \quad (40)$$

we see that the following identity is true:

$$\bar{\nabla}_{[a} C_{bc]|de} + \eta_{d[a} \tilde{C}_{bc]|e} - \eta_{e[a} \tilde{C}_{bc]|d} \equiv 0 , \quad (41)$$

from which, upon taking traces, we obtain

$$\tilde{C}_{ab|c} \equiv \frac{1}{D-3} \bar{\nabla}^d C_{ab|cd} \approx 0 , \quad (42)$$

thereby justifying the identity of the tensors \tilde{C} appearing in (17) and (39). Notice that the relation (33) can easily be derived from (42) and (39).

We can now explain in which sense the field equation (42) can be read as a zero-torsion condition. If one defines, in accordance with (12), the *geometric* — or *dual* — vielbein $\tilde{e}^a(\omega)$ around AdS_D by its components

$$\tilde{e}_\mu^a(\omega) := \bar{e}_\mu^a - \frac{1}{\lambda^2} P_{\mu|}^a + \mathcal{O}(\omega^2) , \quad (43)$$

then the field equations (42) precisely are the expression of the zero-torsion condition for the full spin-connection

$$\nabla = d + w , \quad w := \bar{\omega} + \omega , \quad (44)$$

up to first order in the fluctuation ω :

$$0 \approx T_{\mu\nu}^a := 2 \nabla_{[\mu} \tilde{e}_{\nu]}^a = \bar{T}_{\mu\nu}^a + \frac{1}{\lambda^2} \tilde{C}_{\mu\nu|}^a + \mathcal{O}(\omega^2) , \quad (45)$$

since $\bar{T}_{\mu\nu}^a$, the torsion of AdS, vanishes identically.

We have thus shown that the gauge-invariant completion of the antisymmetrised derivative $\bar{\nabla}_{[b} P_{a]|c}$ is actually null on-shell, whereas the symmetrised covariant derivatives of $\bar{\nabla}_{(a} P_{b)|c}$ are not gauge invariant. From what we have discussed above and the decomposition (29), we are thus led to look at the various contributions of the covariant derivatives of $C_{ab|cd}$, the traceless part of the Riemann-like tensor. We already know that its component $W_{ab|cd}$ is non-vanishing on-shell, and that in fact it is the only component of $C_{ab|cd}$ that does not vanish on-shell, as we showed that both $\hat{J}_{abc|d}$ and \hat{K}_{abcd} are zero on the solutions of (17). Working on-shell, to first order in the derivatives of the $C_{ab|cd}$, we have to consider the two linearly independent contributions $\bar{\nabla}_{[a} W_{bc]|de}$ and $\bar{\nabla}_{(a} W_{bc),de}$, where $W_{ab,cd} := W_{c(a|b)d}$, where we separate groups of symmetrised indices by a coma. It is an important consequence of the identity (41) that, on-shell where the 3 tensors $\tilde{C}_{ab|c}$, $\hat{J}_{abc|d}$ and \hat{K}_{abcd} are vanishing, we have

$$\bar{\nabla}_{[a} W_{bc]|de} \approx 0 . \quad (46)$$

Therefore, only the component $\bar{\nabla}_{(a} W_{bc),de}$ will have to be considered. Acting on the left-hand side of (46) with $\bar{\nabla}^a$ and using the algebraic symmetries of $W_{ab|cd}$ together with the identity

$$[\bar{\nabla}_m, \bar{\nabla}_a] W_{cd|b}{}^m = \lambda^2 (D - 2) W_{ab|cd} + (W_{ac|bd} - W_{ad|bc}) \quad (47)$$

and the on-shell equalities

$$\bar{\nabla}^a W_{ab|cd} \approx \bar{\nabla}^a C_{cd|ab} \approx 0 , \quad (48)$$

we deduce that

$$\begin{aligned} \square W_{bc|ef} + [\bar{\nabla}^a, \bar{\nabla}_b] W_{ca|ef} + [\bar{\nabla}^a, \bar{\nabla}_c] W_{ab|ef} &\approx 0 \\ \Leftrightarrow \\ (\square - 2\lambda^2 (D - 1)) W_{ab|cd} &\approx 0 , \end{aligned} \quad (49)$$

which is the D'Alembert equation characterising a massless spin-2 field freely propagating on AdS_D , as announced in the preamble of the Section. The component $\bar{\nabla}_{(a} W_{bc),de}$ of the first derivative of the primary Weyl tensor is linearly independent from $\bar{\nabla}_{[a} W_{bc]|de}$. It is traceless on-shell, due to $\bar{\nabla}^a W_{ab|cd} \approx 0$, and its traceless part is not constrained by the field equations, being independent from $\bar{\nabla}_{[a} W_{bc]|de}$.

3.5 General structure

Let us generalise what we observed in the previous subsections: Suppose that, after having taken $k-1$ derivatives of $W_{ab|cd}$, the only non-vanishing gauge-invariant projection remaining is $\begin{array}{|c|c|c|} \hline & & k-1 \\ \hline & & \\ \hline \end{array}$, i.e. all $\mathfrak{so}(1, D-1)$ -irreducible projections containing more than 2 rows or more than 2 boxes in the second row are identically zero. Then, applying k symmetrised derivatives on $W_{ab|cd}$ will yield:

$$\begin{array}{|c|} \hline k \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = \begin{array}{|c|c|} \hline & k \\ \hline & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline & & k-1 \\ \hline & & \\ \hline & & \\ \hline \end{array} + \text{traces} \quad (50)$$

Firstly, the second Young diagram contains more than 2 rows and thus vanishes on-shell, as a consequence of (46). Secondly, the trace terms will be zero, up to lower-order terms in the covariant derivatives, by virtue of (48). Together with the equation (49), this finishes the proof that the on-shell degrees of freedom propagated by the pure spin-connection EOM (17) correspond to a massless spin-2 graviton around (A)dS $_D$ background.

4 Toward a nonlinear completion

At the free level, there are in general two parent actions displaying the equivalence, à la Fradkin–Tseytlin [32], of two dual action principles; see [20] in the context of linearised gravity around Minkowski background. In this section, we discuss a possible nonlinear completion of the pure spin-connection action (14). One might speculate that this functional could play the rôle of an alternative parent action, to be considered alongside the first-order MacDowell–Mansouri action.

4.1 Discussion about a nonlinear completion

At this stage, it is tempting to consider the following first-order action

$$S[\omega, e] = -\frac{1}{2\lambda^2} \int_{\mathcal{M}_D} \epsilon_{abcdk_1\dots k_{D-4}} C^{ab} C^{cd} e^{k_1} \dots e^{k_{D-4}}, \quad (51)$$

where this time, $C^{ab}(e, \omega) = R^{ab} - 2e^{[a}P^{b]}$ is the full Weyl 2-form with $P^a = \frac{1}{D-2} \left(R^a - e^a \frac{R}{2(D-1)} \right)$ the complete Schouten 1-form. The Ricci 1-form is defined in terms of the inverse vielbein $E^\mu{}_a$, viz. $R_\mu{}^a = E^\nu{}_b R_{\mu\nu}{}^{ab}$, and the same for the Ricci scalar $R = E^\mu{}_a R_\mu{}^a$. We warn the reader that, although we use the same symbols as in the previous section, the above quantities are the full, non-linear ones. The general context of the discussion in the present section should prevent any confusion.

Despite its formal resemblance of (51) with the frame formulation [33] of Weyl’s gravity, the latter two action principles are inequivalent. In fact, the action principle in [33] is actually of second order in the vierbein as it assumes that the spin connection ω is expressed in terms of the vierbein e via

the condition that the torsion vanishes. To stress again the difference between (51) and Weyl's gravity, notice that, even in $D = 4$, the nonlinear parent action (54) does *not* enjoy the Weyl symmetry

$$\delta_\sigma \omega_\mu{}^{ab} = 2e_\mu{}^{[a} e^{b]\nu} \partial_\nu \sigma, \quad \delta_\sigma e^a = \sigma e^a, \quad (52)$$

since

$$\delta_\sigma C_{\mu\nu}{}^{ab} = 2T_{\mu\nu}{}^{[a} e^{b]\rho} \partial_\rho \sigma. \quad (53)$$

The equation of motion $\frac{\delta S}{\delta e_\mu^\lambda} \approx 0$ for the action (51) are (for $D \geq 4$),

$$\epsilon^{\mu\nu\rho\sigma\tau_1\dots\tau_{D-4}} \epsilon_{abcdk_1\dots k_{D-4}} \left(\delta_\mu^\lambda \delta_m^a R_\nu{}^b C_{\rho\sigma}{}^{cd} e_{\tau_1}^{k_1} + (D-4) \delta_{\tau_1}^\lambda \delta_m^{k_1} C_{\mu\nu}{}^{ab} C_{\rho\sigma}{}^{cd} \right) e_{\tau_2}^{k_2} \dots e_{\tau_{D-4}}^{k_{D-4}} \approx 0. \quad (54)$$

whereas $\frac{\delta S}{\delta \omega_\mu{}^{ab}} \approx 0$ yields:

$$\epsilon^{\mu\nu\rho\sigma\tau_1\dots\tau_{D-4}} \epsilon_{abcdk_1\dots k_{D-4}} \left(\nabla_\nu C_{\rho\sigma}{}^{cd} e_{\tau_1}^{k_1} + 2(D-4) C_{\rho\sigma}{}^{cd} T_{\nu\tau_1}{}^{k_1} \right) e_{\tau_2}^{k_2} \dots e_{\tau_{D-4}}^{k_{D-4}} \approx 0, \quad (55)$$

Notice that the above equation does not imply that the torsion $T^a := de^a + \omega^a{}_b e^b$ vanishes on-shell. In this dual picture where the postulated parent action is (51), there is no reason to expect that the fields that we denoted by e_μ^a should be identified with the geometric vielbein.

We expect that, as in the linearised case studied in the previous section, the field equations for the fields e_μ^a enable one to identify the geometric vielbein with a function of the spin-connection that, around AdS_D , starting with the Schouten tensor, see (43). And similarly to the linearised case, we expect that the field equation (55) for the spin-connection should amount to the vanishing-torsion condition for the geometric vielbein, as we have shown in (45). The torsion T^a for e^a will appear, as for example in the nonlinear generalisation of the identity (41):

$$\nabla C^{ab} + 2T^{[a} P^{b]} + 2e^{[a} \tilde{C}^{b]} \equiv 0, \quad \text{where } \tilde{C}^a := -\nabla P^a. \quad (56)$$

In $D = 4$ dimensions and starting from the action (51), one readily sees that the second term in the field equations for the vielbein is absent so that the latter reduce to

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} P_\nu{}^b C_{\rho\sigma}{}^{cd} \approx 0 \quad (D = 4). \quad (57)$$

On top of the trivial solution $C_{\mu\nu}{}^{ab} = 0$ which covers the conformally flat spaces, the above equation also admits an obvious class of solutions, namely those of the type

$$P_\mu{}^a = \frac{1}{2(D-1)f(x)} e_\mu{}^a \Leftrightarrow R_\mu{}^a = \frac{1}{f(x)} e_\mu{}^a, \quad (58)$$

for any (smooth) nowhere vanishing function $f(x)$. This class of solutions includes Einstein-like⁸ spaces for constant functions $f(x) = k \neq 0$, and, *a fortiori*, maximally symmetric spaces.

⁸We use this terminology, in accordance with the linearised analysis made above, to stress that the torsion T^a is not necessarily zero.

Notice that these two classes of solutions are not disjoint. More precisely, in the case where the torsion T^a is zero, a spacetime is both conformally flat and Einstein if and only if it is of constant curvature (since the Ricci scalar is the only nonvanishing Lorentz-irreducible component of the Riemann curvature, in which case the Ricci scalar must be constant by virtue of the Bianchi identity). In this respect, maximally-symmetric spacetimes such as (A)dS are the simplest spaces in the intersection between these two classes.

Disregarding the conformally flat solutions of the EOM (57) which do not fall into the class $R_\mu{}^a = \frac{1}{f(x)} e_\mu{}^a$, we can solve perturbatively, around AdS_D , the equation (57) for the vierbein in terms of the spin-connection. Of course, one cannot have the solution in closed form, but only order by order. At the lowest order in expansion and for the function $f = -\frac{1}{(D-1)\lambda^2} = \frac{1}{2\Lambda/(D-2)}$, *i.e.* the constant corresponding to AdS_D , the action (51) reproduces (14). It seems that the cubic part of the pure spin-connection action thereby obtained by substituting the vierbein in terms of the spin-connection in $S[\omega, e]$ reproduces the result presented in [8].

On the branch (58) with $P_\mu{}^a = -\lambda^2/2 e_\mu{}^a$ that, in particular, contains AdS_D , we have that the fully nonlinear child action $S[\omega] = S[\omega, e(\omega)]$ assumes exactly the same value as the child action of the MacDowell–Mansouri action (1). Indeed, in the present case the Weyl two-form becomes $C^{ab} = R^{ab} + \lambda^2 e^a e^b$ which coincides with the AdS_D curvature \mathcal{R}^{ab} .

4.2 Adding the Pontryagin invariant

In this subsection we adopt the Euclidean signature and add a topological invariant to the parent action (51) in $D = 4$, the Hirzebruch signature $\tau(\mathcal{M}_4)$ of the manifold:

$$\tau(\mathcal{M}_4) = \frac{1}{3} \int_{\mathcal{M}_4} p_1(\mathcal{M}_4), \quad (59)$$

where $p_1(\mathcal{M}_4)$ the first Pontryagin class which, in this particular case, can be written as:

$$p_1(\mathcal{M}_4) = -\frac{1}{8\pi^2} \epsilon_{abcd} R^{ab} \wedge *R^{cd} = -\frac{1}{8\pi^2} \epsilon_{abcd} C^{ab} \wedge *C^{cd}, \quad (60)$$

where $*$ denotes the Hodge dual. Using the identity in Lorentzian Euclidean signature

$$\epsilon_{abcd} C^{ab} \wedge C^{cd} = \epsilon_{abcd} *C^{ab} \wedge *C^{cd}, \quad (61)$$

the action (51) in $D = 4$ can be rewritten as $S[e, \omega] = -\frac{1}{4\lambda^2} \int_{\mathcal{M}_4} \epsilon_{abcd} (C^{ab} \wedge C^{cd} + *C^{ab} \wedge *C^{cd})$.

By adding to it a term proportional to the Hirzebruch invariant, we can recast the resulting action in

a form that only depends on the (anti-) self dual part of the Weyl tensor. More precisely, one has

$$\begin{aligned}
S^\tau[e, \omega] &= -\frac{1}{4\lambda^2} \int_{\mathcal{M}_4} \epsilon_{abcd} \left(C^{ab} \wedge C^{cd} + *C^{ab} \wedge *C^{cd} \right) \pm \frac{12\pi^2}{\lambda^2} \tau(\mathcal{M}_4) \\
&= -\frac{1}{4\lambda^2} \int_{\mathcal{M}_4} \epsilon_{abcd} \left(C^{ab} \wedge C^{cd} + *C^{ab} \wedge *C^{cd} \pm \left[C^{ab} \wedge *C^{cd} + *C^{ab} \wedge C^{cd} \right] \right) \\
&= -\frac{1}{4\lambda^2} \int_{\mathcal{M}_4} \epsilon_{abcd} \left(C^{ab} \pm *C^{ab} \right) \wedge \left(C^{cd} \pm *C^{cd} \right) \\
&= -\frac{2}{\lambda^2} \int_{\mathcal{M}_4} e d^4x \sqrt{\det(C^{ab} \pm *C^{ab})}
\end{aligned} \tag{62}$$

where the last line is obtained by recalling the relation $\det(A) = [\text{Pf}(A)]^2$ and after a small abuse of notation, by extracting the volume form out of the wedge product of the 2-forms appearing on the second to last line.

A relation between the MacDowell–Mansouri action and Ashtekar’s formulation with cosmological term was given in [34]. For discussions about MacDowell–Mansouri formulation of gravity in the context of S-duality, see for instance [35, 36, 17] and refs. therein. It would be interesting to perform similar analyses starting from the alternative action (62), where the difference is essentially that the Weyl two-form replaces the (A)dS_D curvature two-form. It would also be interesting to reconsider the work [37] starting from the action (62).

5 Conclusions

When the spin-connection is dynamical and the cosmological constant is nonvanishing, the vielbein becomes an auxiliary field in the technical sense that it can be integrated out via its own algebraic equation of motion. In the present paper, we analysed in details at quadratic level the corresponding pure spin-connection formulation of GR with a cosmological constant arising from the first-order MacDowell–Mansouri action.

We proved, using the unfolded technology, that the fluctuations of the dynamical spin-connection around (A)dS_D propagate a massless spin-two particle. We have also shown, by going to the traceless gauge, that the Hodge dual of the gauge field is a $GL(D)$ -irreducible tensor field in the same representation as the dual massive graviton on flat spacetime of the same dimension. Finally, the first-order quadratic action has been rewritten suggestively as the square of the linearised Weyl-like two-form. Interestingly, the nonlinear completion of this action obtained by inserting the full Weyl-like two-form is distinct from MacDowell–Mansouri’s and Weyl’s actions, though intimately related to both of them since they share the same values on Einstein spaces.

The nonlinear extension of the electric-magnetic duality of linearised gravity remains an important challenge since, by analogy with its spin-one counterpart, such a duality should relate weak and strong coupling regimes. In the presence of a cosmological constant, another type of duality is available between the conventional descriptions of gravity and exotic pure-connection descriptions. One can see on dimensional ground that the loop expansion in any pure-connection formulation of gravity (Eddington–Schrödinger’s or pure spin-connection) is controlled by the ratio of the Planck length over the cosmological scale $|\Lambda|^{-\frac{1}{2}}$. As exhibited above at linearised level, these two types of dualities are deeply intertwined, if not even two faces of the same coin. Put together, these remarks suggest the existence of two dual descriptions of gravity with, respectively, small and large values of the cosmological constant.

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